COE 755

**Queueing Theory**

Spring 2013

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Exam Preparation Problems

(Probability theory + Random variables)

# I. Probability theory

# Problem I

The probability that a train leaves on time is 0.85. The probability that it leaves on time and arrives on time is 0.6. If the train leaves on time, what is the probability that it will arrive on time.

# Solution

Let A be the event “leaves on time” and let B be the event “arrives on time”. We are given P(A) = 0.85 and . Then

# Problem II

Flip a coin. If the coin comes up heads, select one ball from jar A. If the coin comes up tails, select one ball from jar B. Jar A contains 8 red and 2 green balls. Jar B contains 15 red and 30 green balls. Find P[red | head] and P[red | tails]

# Solution

P[red | heads] should be interpreted as the probability of getting a red ball, given that the coin flip was a head. This means that the ball is to be selected from jar A. As such, you can rephrase the question as P[red | A] = 8/(8+2) = 0.8. Similarly,

P[red | tails] = P[red | B] = 15/(15+30) = 1/3 = 0.333…

# Problem III

A grab bag contains 20 “prizes” in identical boxes. Of the “prizes”,

12 have a value of $ 2

6 have a value of $ 5

1 has a value of $ 10

1 has a value of $ 20

What is your mathematical expectation if you select one of the “prizes”? How would you feel about paying $8 to get to select one of these?

# Solution

Given that we have 20 tickets, then:

The probability is 0.6 that you get $2

The probability is 0.3 that you get $5

The probability is 0.05 that you get $10

The probability is 0.05 that you get $20

The mathematical expectation is then found as:

0.6 x $2 + 0.3 x $5 + 0.05 x $10 + 0.05 x $20 = $4.20, and so it seems silly to play this game.

# II. Binomial random variables

# Problem I

Suppose that you are rolling a die eight times. Find the probability that the face with two spots comes up exactly twice.

# Solution

Let X be the number of successes, meaning the number of times that the face with two spots comes up. This is a binomial situation with n = 8 and p=1/6. The probability of exactly two successes is . This can be done with a calculator. There are various strategies to organize the arithmetic, but the answer certainly comes out as about 0.260476.

# Problem II

The probability of winning at a certain game is 0.10. If you play the game 10 times, what is the probability that you win at most once?

# Solution

Let X be the number of winners. This is a binomial situation with n = 10 and p = 0.10. We interpret “win at most once” as “X ≤1”. Then

# Problem III

If X is binomial with parameters n and p, find an expression for P[X≤1]

# Solution

This is the same as the previous problem, but it is in a generic form.

# Problem IV

The probability is 0.038 that a person reached on a “cold call” by a telemarketer will make a purchase. If the telemarketer calls 40 people, what is the probability that at least one sale will result?

# Solution

Let X be the resulting number of sales. Certainly X is binomial with n = 40, and p=0.038. This “at least one” problem can be done with a standard trick:

# Problem V

If you flip a fair coin 19 times, what is the probability that you will end up with an even number of heads?

# Solution

Let X be binomial with n=19 and p=1/2. This seems to be asking for P[X=0] + P[X=1] + … + P[X=18], which is an annoying calculation. However, we have got a trick. Consider the first 18 flips. The cumulative number of heads will either be even or odd. If it is even, then the 19th flip will preserve the even total with probability ½. If it is odd, then the 19th flip will convert it to even with probability ½. At the end our probability of having an even number of heads must be ½.

# III. Poisson random variables

# Problem I

The number of calls arriving at the Swampside Police Station follows a Poisson distribution with rate 4.6/hour. What is the probability that exactly six calls will come between 8:00 p.m. and 9:00 p.m.?

# Solution

Let X be the random number arriving in this one-hour time period. We will use λ=4.6 and then find P[X=6]=0.1323

# Problem II

In the situation above, find the probability that exactly 7 calls will come between 9:00 p.m. and 1:30 p.m.

# Solution

Let Y be the random number arriving during this 90-minute period. The Poisson rate parameter expands and contracts appropriately, so the relevant value of λ is 1.5 x 4.6 = 6.9. We find P[Y=7] = 0.1489

# Problem III

Jobs to be performed on a machine arrive according to a Poisson distribution with a rate of two per hour. Suppose that the machine breaks down from time to time and it takes 1 hour to be repaired. What is the probability that a) zero, b) two, and c) five new jobs will arrive during this time?

# Solution

1. The arrival of jobs is a Poisson distribution with rate λ=2/hr. The probability to have k = 0 jobs arriving within the interval t = 1 hr during which the machine is down:
2. Similarly to the above, we get the probability to have k = 2 jobs arriving within the interval t = 1hr: P[k=2] = 2e-2
3. Finally, the probability to have k = 5 jobs arriving within the interval t=1hr:

# IV. Exponential random variables

# Problem I

Calculate the pdf of the maximum of two exponential distributions with parameters λ1 and λ2. What is the expression of the pdf when λ1 = λ2 = λ.

# Solution

Using the definition of the CDF:

Now, by the definition of the pdf:

If then:

# V. Expected value, variance, covariance, and correlation

# Problem I

Let X and Y be two random variables whose joint probability density function is

fX,Y (x, y) = x + y, if 0 < x < 1 and 0 < y < 1, otherwise, it takes the value 0.

Find E[XY ], Cov[X, Y ], Corr[X, Y ], E[X+Y] and Var[X + Y ].

# Solution

