AMERICAN UNIVERSITY OF BEIRUT
Faculty of Arts and Sciences
Mathematics Department

MATH-CMPS 281
FINAL EXAMINATION
SPRING 2005
Closed Book. Two hours

SUBMIT THE QUESTION SHEET WITH SCRATCH BOOKLET

STUDENT NAME	
ID NUMBER	

Problem	Grade	Out of
1		22
2		24
3		22
4		32
TOTAL		100

1. (22 points) Let v be a non-zero column vector of dimension n.

   (a) (4 points) Give the number of references and the number of flops for the BLAS operations : $v^T v$ and vv^T. Specify the names of these operations.

   (b) (4 points) Let $u = \frac{1}{(v^T v)^{1/2}} v$. Show that $||u|| := ||u||_2 = 1$.

   (c) (4 points) Let $x \in \mathbb{R}^n$ be a column vector. Show that $(uu^T)x = (u^T x)u$. Give the number of flops necessary to compute $(uu^T)x$ without computing uu^T.

   (d) (4 points) Show that the matrix $H = I - 2uu^T$ is symmetric and orthogonal. (where I is the identity n by n matrix).

   (e) (6 points) For $x \in \mathbb{R}^{n,1}$, prove that the GAXPY operation Hx reduces to a SAXPY operation.

2. (24 points) Let $A \in \mathbb{R}^{n,n}$ be a square matrix, such that : (i) A is invertible and (ii) $A = LU$, where L is an $n \times n$ lower triangular matrix with 1's on its diagonal and U is upper triangular $n \times n$.

(a) (6 points) Show that the LU decomposition of A is **unique**.

(b) (6 points) Let D be the diagonal matrix which elements are $u_{ii}|i = 1, ..., n$, the diagonal elements of U. Show that $A = LU$ is equivalent to $A = LDM^T$, where M is a lower triangular matrix with 1's on the diagonal. Give an explicit formula for M.

(c) (6 points) Complete the following MATLAB program that gives the LDM^T decomposition for an invertible matrix A.

```
function [L,D,M]=ldm(A)
% A is invertible and admits the LU decomposition
[L,U]=lu(A);
D=...............;
M=...............;
```

(d) (6 points) Assume we add the assumption that A is symmetric, i.e. $A = A^T$. Prove that $A = LDL^T$, i.e. $M = L$.

3. (22 points) Let $D_n = \{(x_i, y_i) | i = 1, 2, ..., n\}$, with $x_i \neq x_j$, $i \neq j$. Define the **column** vectors of length n $e = (1\,1\,1\,....1)^T$, $x = (x_1\,x_2\,....x_n)^T$ and $y = (y_1\,y_2\,....y_n)^T$. Finally let A be the n by 2 matrix :

$$A = [x\ e]$$

(a) (6 points) Prove that $rank(A) = 2$.

(b) (6 points) Show that the 2 by 2 matrix $B = A^T A$ is **symmetric and positive definite**.

(c) (10 points) The MATLAB commands `w=A\y` and `w=B\(A'*y)` give the same result which the unique solution of the least squares problem

$$||y - Aw||_2 = \min_{z \in \mathbb{R}^2} ||y - Az||_2$$

Specify the equation verified by w then give numerical linear algebra methods that are used by each of these commands.

4. (32 points) Given the matrices A and $B = A^T A$ defined in the previous problem from the data D_n, answer the following :

(a) (4 points) Let $\{\sigma_1, w_1\}$ and $\{\sigma_2, w_2\}$ be the 2 pairs of eigenvalues-eigenvectors associated with the matrix B, with $\sigma_1 \leq \sigma_2$. Using appropriately the MATLAB command [W,D]=eig(B), generate from this command $\{\sigma_1, w_1\}$ and $\{\sigma_2, w_2\}$ by completing the following segment of MATLAB code :

```
[W,D]=eig(B);
sigma1=...........;
sigma2=...........;
w1=...............;
w2=...............;
```

(b) (4 points) State the theorem that assesses why $w_1^T w_2 = 0$ and $w_i^T w_i = 1$, $i = 1, 2$. Also, say why $\sigma_i > 0$, $\forall i = 1, 2$.

(c) (6 points) Let $v_i = \frac{1}{\sqrt{\sigma_i}} A w_i$, $i = 1, 2$. Show that $v_i \in \mathbb{R}^n$, $i = 1, 2$ with $v_1^T v_2 = 0$ and
$v_i^T v_i = 1$, $i = 1, 2$.

(d) (4 points) Prove also that v_1 and v_2 verify :

$$AA^T v_i = \sigma_i v_i, \; i = 1, 2.$$

(e) (4 points) Let $u_i = \frac{1}{\sqrt{\sigma_i}} A^T v_i \in \mathbb{R}^2$, $i = 1, 2$. Prove that

$$AA^T v_i = \sigma_i v_i,\ i = 1, 2.$$

is equivalent to .

$$Au_i = \sqrt{\sigma_i} v_i,\ i = 1, 2.$$

(f) (6 points) Put the equations

$$Au_i = \sqrt{\sigma_i} v_i,\ i = 1, 2,$$

in matrix form. For that purpose, define the matrices $U = [u_1\ u_2]$, $V = [v_1\ v_2]$ and S the 2 by 2 diagonal matrix whose elements are $\sqrt{\sigma_1}$ and $\sqrt{\sigma_2}$. Give the sizes of the matrices U and V, then show that :

$$AU = VS.$$

(g) (4 points) Give the value of $||A||_2$ in terms of σ_1 and σ_2.