MATHEMATICS 218
FALL SEMESTER 2010-2011
FINAL EXAMINATION

Time: 120 minutes Date: January 31, 2011

Name:————————————————————-

ID Number:————————————————————

Section:————————————————————

QUESTION	GRADE
1	/25
2	/20
3	/25
4	/30
5	/30
6	/40
7	/30
TOTAL GRADE	/200

Answer the following set of questions on the allocated pages; the back of pages may be used if needed

1. Let $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

(a) Find the eigenvalues of A. (9 points)

(b) Find a basis for each of the eigenspaces. (10 points)

(c) Deduce that A is diagonalizable and write a formula calculating A^5. (6 points)

2. Consider the following system:

$$\begin{cases} x + 2z = 2 \\ 2x + 4z = 6 \\ y + 2z = 4 \end{cases}$$

(a) Find the least squares solution of the above system $Ax = b$

(12 points)

(b) Find the projection of b onto the Column Space of (A). (8 points)

3. Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in R^4 \mid w + 2x + 2y + 4z = 0 \right\}$

(a) Find a basis for W. (7 points)

(b) Find a basis for $W^\perp$. (8 points)

(c) Use parts (a) and (b) to find an orthogonal basis for R^4 with respect to the Euclidean inner product. (10 points)

4. Let $T : P_3 \to P_3$ be the linear transformation defined by:

$$T(p(x)) = p'(x) - p"(x) + p(0)$$

(a) Let $B = \{1, x, x^2, x^3\}$, find $[T]_B$ (6 points)
(b) Find a basis for the nullspace of T. (8 points)
(c) Find a basis for the range of T. (6 points)
(d) Let $B' = \{1, x - 1, x^2 - 2x, x^3\}$, use (a) to find $[T]_{B'}$.
(10 points)

5. **Answer the following INDEPENDENT questions.**

(10 points each)

(a)

- For which values of t is $< p, q >= \int_t^2 p(x)q(x)dx$ an inner product on $V = P_2$?
- For which of the values of t obtained above would the two polynomials $p(x) = x + 1$ and $q(x) = 2$ be orthogonal?

(b) Let W be a subspace of $V = M_{2\times 2}$, with basis $B = \{I, A\}$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}$.
Let $T : W \to W$ be given by $T(M) = AM$.

- Find $[T]_B$
- Use $[T]_B$ to compute $T(3I - A)$.

(c) For which value(s) of k would the vector $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ be an eigenvector of the matrix $A = \begin{bmatrix} k & 0 \\ 1 & -1 \end{bmatrix}$?

6. **Prove the following:** (10 points each)

(a) Let $W = span\,\{w\}$ be a subspace of an inner product space V. Let $b \in V$ and not in W. Use the Cauchy-Schwartz inequality to prove that:

$$||proj_W(b)|| \leq ||b||$$

(b) Let $V = P_2$ and $p, q \in P_2$. Show that:

$$< p, q >= p(0)q(0) + p(\frac{1}{2})q(\frac{1}{2}) + p(1)q(1)$$

is an inner product in P_2.

(c) Let A be an $n \times n$ matrix.

- Show that if $A^2 = 0$ then Column space of $(A) \subset$ Nullspace (A)
- Deduce that if $A^2 = 0$ then $rankA \leq \frac{n}{2}$

(d) Let $T : V \to V$ be a linear transformation such that $T \circ T = T$. Show that $Range(T) \cap Nullspace(T) = 0$

7. Indicate whether each of the following statements is TRUE (T) or FALSE (F) without justifying your answer. (3 points each)

—-(a) The set of all 3×3 matrices with $det(A) = 0$ is a subspace of $M_{3\times 3}$.

—-(b) Let $T : P_2 \to R$ defined by $T(ax^2 + bx + c) = |a|$ is a linear transformation.

—-(c) Let $T : V \to W$ be a linear transformation. If $dimV < dimW$ then T cannot be onto.

—-(d) Let A, B and C be $n \times n$ matrices, if $AB = AC$ then $B = C$.

—-(e) The dimension of the vector space of the $n \times n$ upper triangular matrices is $\frac{n^2-n}{2}$.

—-(f) $T : V \to W$ be a linear transformation with $\{v_1 \dots v_n\}$ linearly independent in V, then $\{T(v_1) \dots T(v_n)\}$ are linearly independent.

—-(g) If λ is an eigenvalue of an idempotent matrix A (i.e: $A^2 = A$) then $\lambda = 0$ or $\lambda = 1$.

—-(h) In R^2 with the standard inner product, the orthogonal complement of $y = 2x$ is $y = \frac{1}{2}x$.

—-(i) If $T : R^n \to R^m$ is a linear transformation and $m = n$ then T is an isomorphism.

—-(j) If λ is an eigenvalue for A then λ^n is an eigenvalue for A^n for all $n \in \mathbb{Z}$.