

# CHAPTER 23

## GAUSS' LAW

23-1

# INTRODUCTION

## Chapter 22: Electric field of an object

- We split the charge distribution up into charge elements  $dq$ ,
- Found the field due to an element, and resolved the vector into components
- Then we determined whether the components from all the elements would end up canceling or adding.
- Finally we summed the adding components by integrating over all the elements, with several changes in notation along the way.



Carl Friedrich Gauss (1777–1855)

- One of the primary goals of physics is to find simple ways of solving such labor-intensive problems.
- We can save far more work by using Gauss' law
- Gauss' law is an alternative method to calculate the electric field
- Gauss' law considers a hypothetical (imaginary) closed surface (**Gaussian surface**) enclosing the charge distribution
- The Gaussian surface mimics the symmetry of the charge distribution

The problems in this chapter are of two types:

- Sometimes we know the charge and we use Gauss' law to find the field at some point.
- Sometimes we know the field on a Gaussian surface and we use Gauss' law to find the charge enclosed by the surface.

We need a quantitative way of determining how much electric field pierces a surface. That measure is called the electric flux.

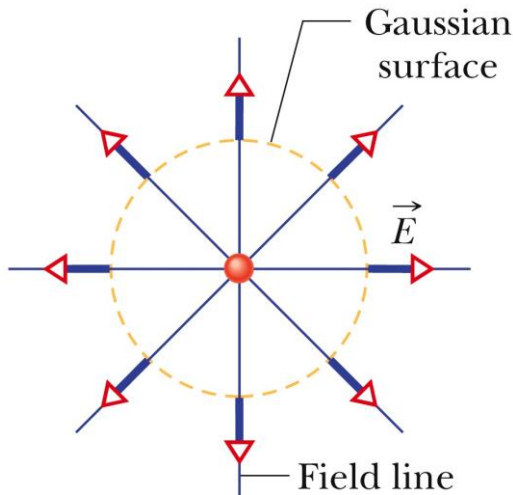
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# Electric Flux

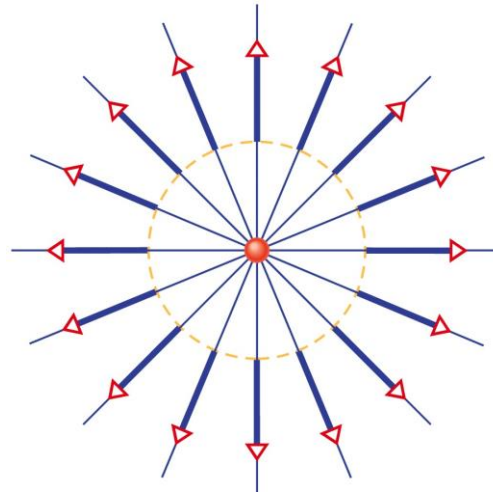
## 23-1 Electric Flux



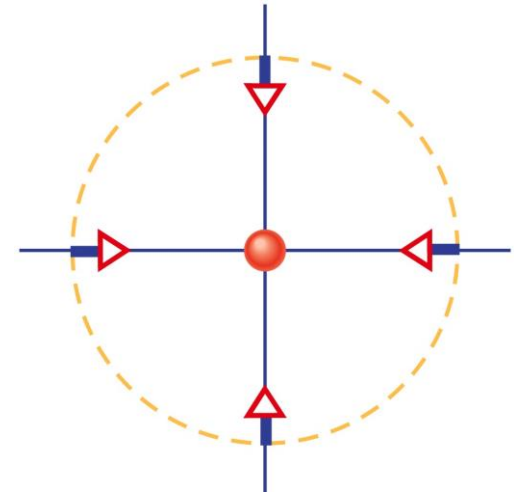
Gauss' law relates the electric field at points on a (closed) Gaussian surface to the net charge enclosed by that surface.



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Electric field vectors and field lines pierce an imaginary, spherical Gaussian surface that encloses a particle with charge  $+Q$ .

Now the enclosed particle has charge  $+2Q$ .

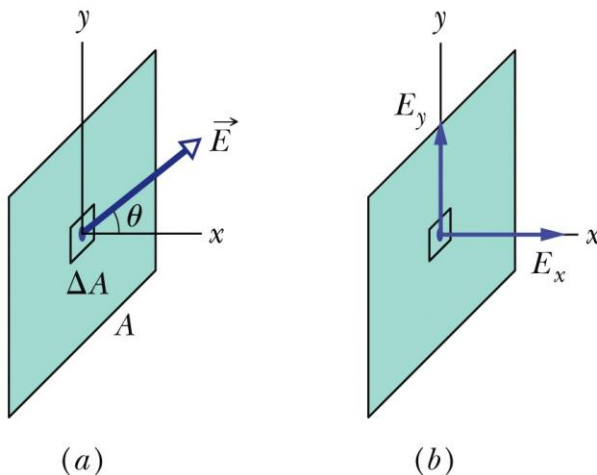
Can you tell what the enclosed charge is now?  
*Answer:  $-0.5Q$*

## 23-1 Electric Flux

The **area vector**  $d\mathbf{A}$  for an area element (patch element) on a surface is a vector that is perpendicular to the element and has a magnitude equal to the area  $dA$  of the element.

The **electric flux**  $d\Phi$  through a patch element with area vector  $d\mathbf{A}$  is given by a dot product:

$$d\Phi = \vec{E} \cdot d\vec{A}.$$



- (a) An electric field vector pierces a small square patch on a flat surface.
- (b) Only the  $x$  component actually pierces the patch; the  $y$  component skims across it.

$$E_x = E \cos \theta$$

$$\Delta\Phi = (E \cos \theta) \Delta A.$$

The SI unit of flux is the newton–square-meter per coulomb ( $\text{N} \cdot \text{m}^2/\text{C}$ )

## 23-1 Electric Flux

Now we can find the total flux by integrating the dot product over the full surface.

The **total flux** through a surface is given by

$$\Phi = \int \vec{E} \cdot d\vec{A} \quad (\text{total flux}).$$

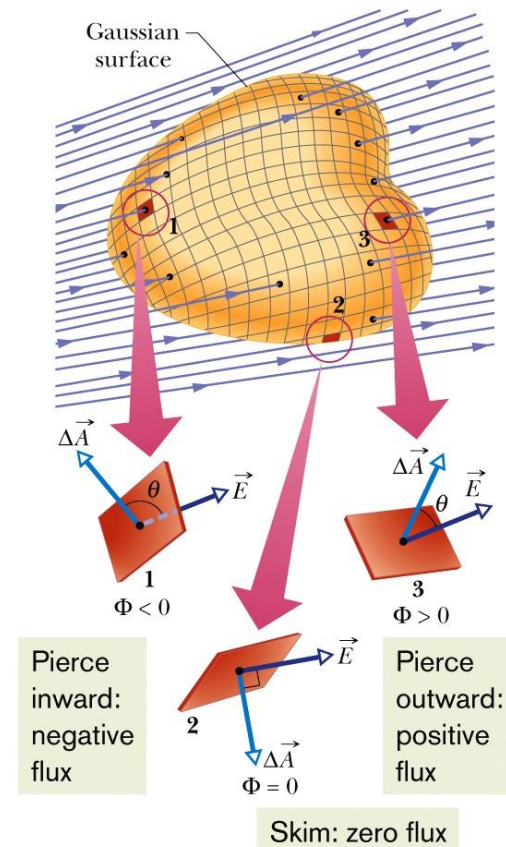
The **net flux** through a closed surface (which is used in Gauss' law) is given by

$$\Phi = \oint \vec{E} \cdot d\vec{A} \quad (\text{net flux}).$$

where the integration is carried out over the entire surface.



An inward piercing field is negative flux. An outward piercing field is positive flux. A skimming field is zero flux.



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# 23-1 Electric Flux

## Flux through a closed cylinder, uniform field

Figure 23-6 shows a Gaussian surface in the form of a closed cylinder (a Gaussian cylinder or G-cylinder) of radius  $R$ . It lies in a uniform electric field  $\vec{E}$  with the cylinder's central axis (along the length of the cylinder) parallel to the field. What is the net flux  $\Phi$  of the electric field through the cylinder?

### KEY IDEAS

We can find the net flux  $\Phi$  with Eq. 23-4 by integrating the dot product  $\vec{E} \cdot d\vec{A}$  over the cylinder's surface. However, we cannot write out functions so that we can do that with one integral. Instead, we need to be a bit clever: We break up the surface into sections with which we can actually evaluate an integral.

**Calculations:** We break the integral of Eq. 23-4 into three terms: integrals over the left cylinder cap  $a$ , the curved cylindrical surface  $b$ , and the right cap  $c$ :

$$\begin{aligned}\Phi &= \oint \vec{E} \cdot d\vec{A} \\ &= \int_a \vec{E} \cdot d\vec{A} + \int_b \vec{E} \cdot d\vec{A} + \int_c \vec{E} \cdot d\vec{A}. \quad (23-5)\end{aligned}$$

Pick a patch element on the left cap. Its area vector  $d\vec{A}$  must be perpendicular to the patch and pointing away from the interior of the cylinder. In Fig. 23-6, that means the angle between it and the field piercing the patch is  $180^\circ$ . Also, note that the electric field through the end cap is uniform and thus  $E$  can be pulled out of the integration. So, we can write the flux through the left cap as

$$\int_a \vec{E} \cdot d\vec{A} = \int_a E(\cos 180^\circ) dA = -E \int_a dA = -EA,$$

where  $\int_a dA$  gives the cap's area  $A (= \pi R^2)$ . Similarly, for the right cap, where  $\theta = 0$  for all points,

$$\int_c \vec{E} \cdot d\vec{A} = \int_c E(\cos 0) dA = EA.$$

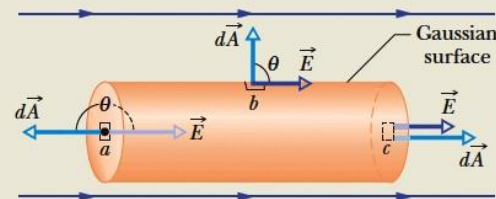
Finally, for the cylindrical surface, where the angle  $\theta$  is  $90^\circ$  at all points,

$$\int_b \vec{E} \cdot d\vec{A} = \int_b E(\cos 90^\circ) dA = 0.$$

Substituting these results into Eq. 23-5 leads us to

$$\Phi = -EA + 0 + EA = 0. \quad (\text{Answer})$$

The net flux is zero because the field lines that represent the electric field all pass entirely through the Gaussian surface, from the left to the right.



**Figure 23-6** A cylindrical Gaussian surface, closed by end caps, is immersed in a uniform electric field. The cylinder axis is parallel to the field direction.

23-2

# GAUSS' LAW

## 23-2 Gauss' Law

Gauss' law relates the net flux  $\Phi$  of an electric field through a closed surface (a Gaussian surface) to the *net* charge  $q_{enc}$  that is enclosed by that surface. It tells us that

$$\epsilon_0 \Phi = q_{enc} \quad (\text{Gauss' law}).$$

we can also write Gauss' law as

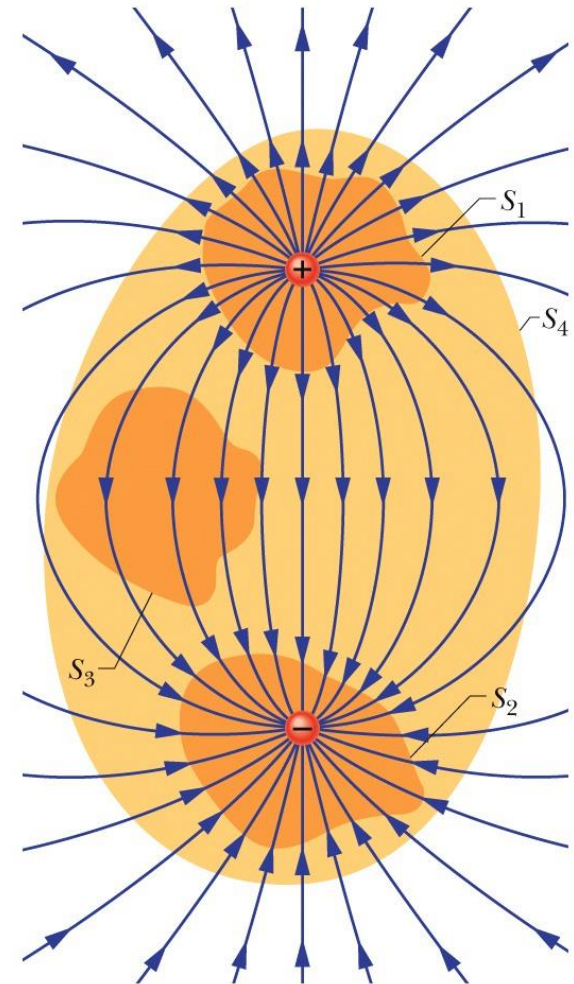
$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc} \quad (\text{Gauss' law}).$$

## 23-2 Gauss' Law

Two charges, equal in magnitude but opposite in sign, and the field lines that represent their net electric field. Four Gaussian surfaces are shown in cross section.

**Surface S1.** The electric field is outward for all points on this surface. Thus, the flux of the electric field through this surface is positive, and so is the net charge within the surface, as Gauss' law requires

**Surface S2.** The electric field is inward for all points on this surface. Thus, the flux of the electric field through this surface is negative and so is the enclosed charge, as Gauss' law requires.



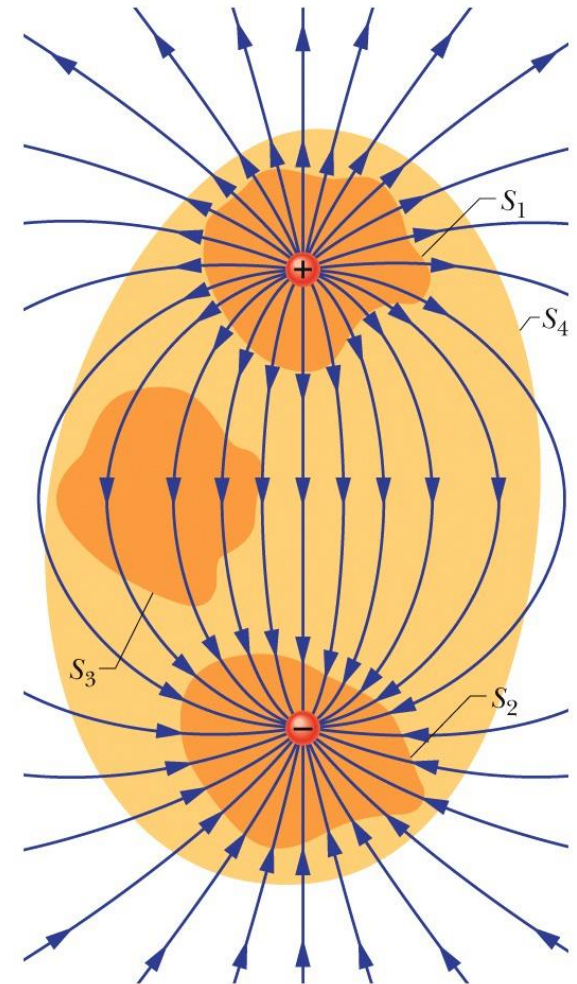
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## 23-2 Gauss' Law

Two charges, equal in magnitude but opposite in sign, and the field lines that represent their net electric field. Four Gaussian surfaces are shown in cross section.

**Surface S3.** This surface encloses no charge, and thus  $q_{enc} = 0$ . Gauss' law requires that the net flux of the electric field through this surface be zero. That is reasonable because all the field lines pass entirely through the surface, entering it at the top and leaving at the bottom.

**Surface S4.** This surface encloses no net charge, because the enclosed positive and negative charges have equal magnitudes. Gauss' law requires that the net flux of the electric field through this surface be zero. That is reasonable because there are as many field lines leaving surface S4 as entering it.



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# 23-3

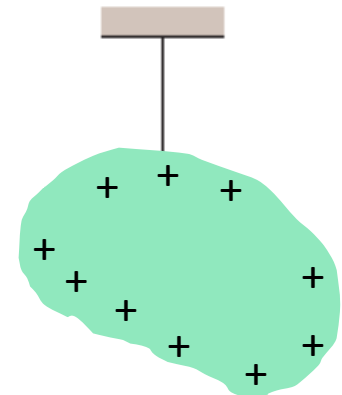
## **A Charged Isolated Conductor**

## 23-3 A Charged Isolated Conductor



If an excess charge is placed on an isolated conductor, that amount of charge will move entirely to the surface of the conductor. None of the excess charge will be found within the body of the conductor.

- The electric field inside a conductor is zero
- Gauss' law is used to find the electric field magnitude  $E$  near an isolated conducting surface with a uniform surface charge density  $\sigma$ .



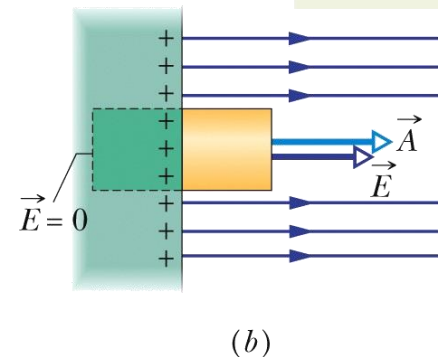
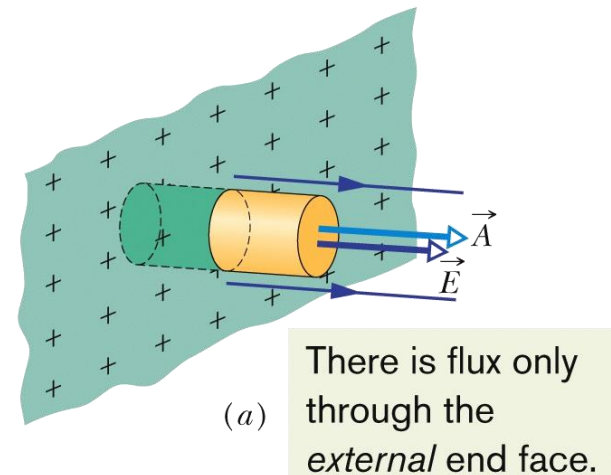


## 23-3 A Charged Isolated Conductor

$$E = \frac{\sigma}{\epsilon_0} \quad (\text{conducting surface}).$$

(a) Perspective view

(b) Side view of a tiny portion of a large, isolated conductor with excess positive charge on its surface. A (closed) cylindrical Gaussian surface, embedded perpendicularly in the conductor, encloses some of the charge. Electric field lines pierce the external end cap of the cylinder, but not the internal end cap. The external end cap has area  $A$  and area vector  $\mathbf{A}$ .



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# 23-4

## **Applying Gauss' Law: Cylindrical Symmetry**

## 23-4 Applying Gauss' Law: Cylindrical Symmetry

Figure shows a section of an **infinitely long cylindrical plastic rod with a uniform charge density  $\lambda$** . The charge distribution and the field have cylindrical symmetry. To find the field at radius  $r$ , we enclose a section of the rod with a concentric Gaussian cylinder of radius  $r$  and height  $h$ . The net flux through the cylinder from Gauss' Law reduces to

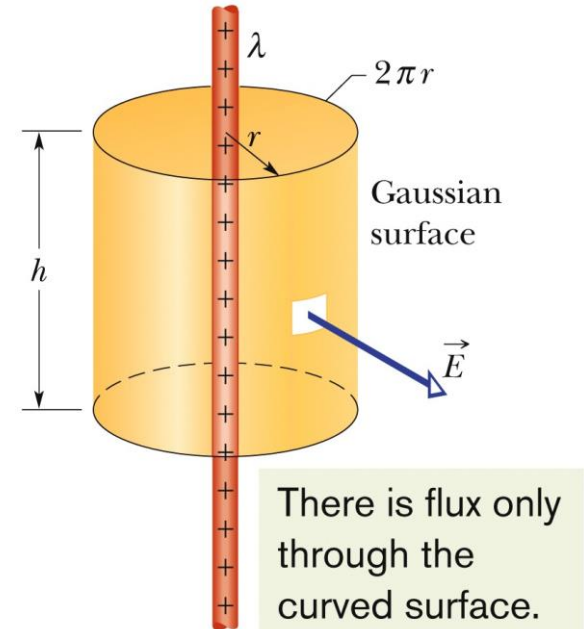
$$\Phi = EA \cos \theta = E(2\pi rh) \cos 0 = E(2\pi rh).$$

yielding

$$\epsilon_0 \Phi = q_{\text{enc}},$$

$$\epsilon_0 E(2\pi rh) = \lambda h,$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \quad (\text{line of charge}).$$



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A Gaussian surface in the form of a closed cylinder surrounds a section of a very long, uniformly charged, cylindrical plastic rod.

23-5

## **Applying Gauss' Law: Planar Symmetry**

## 23-5 Applying Gauss' Law: Planar Symmetry

### Non-conducting thin Sheet

Figure (a-b) shows a portion of a thin, infinite, non-conducting sheet with a uniform (positive) surface charge density  $\sigma$ . A sheet of thin plastic wrap, uniformly charged on one side, can serve as a simple model.

Gauss' Law,

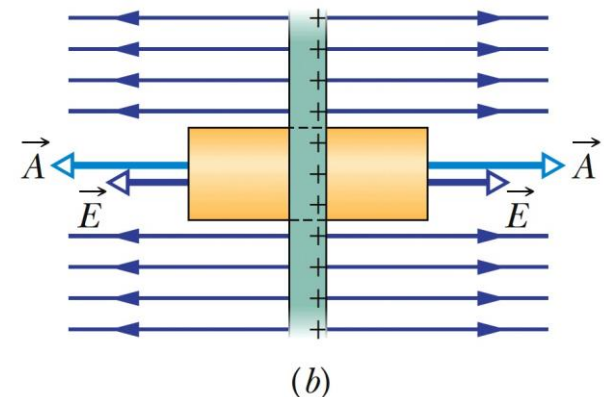
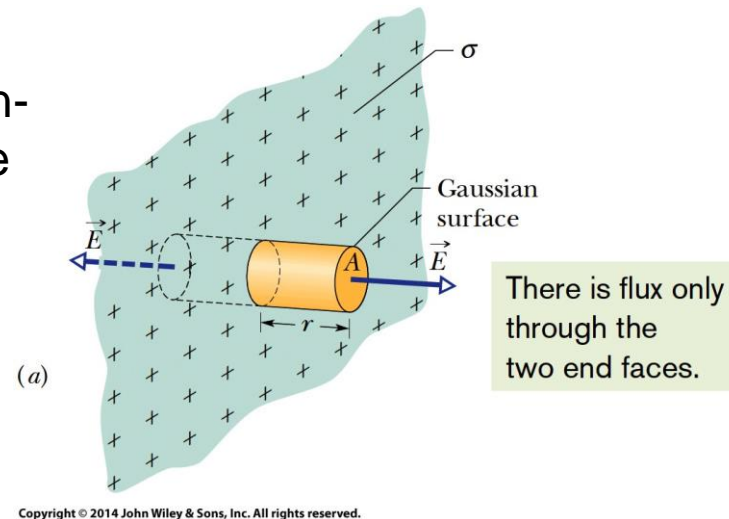
$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{\text{enc}},$$

becomes

$$\epsilon_0(EA + EA) = \sigma A,$$

where  $\sigma A$  is the charge enclosed by the Gaussian surface. This gives

$$E = \frac{\sigma}{2\epsilon_0} \quad (\text{sheet of charge}).$$



## 23-5 Applying Gauss' Law: Planar Symmetry

### Two conducting Plates

**Figure (a)** shows a cross section of a thin, infinite conducting plate with excess positive charge. **Figure (b)** shows an identical plate with excess negative charge having the same magnitude of surface charge density  $\sigma$ .

Suppose we arrange for the plates of **Figs. a** and **b** to be close to each other and parallel (**c**). Since the plates are conductors, when we bring them into this arrangement, the excess charge on one plate attracts the excess charge

on the other plate, and all the excess charge moves onto the inner faces of the plates as in Fig.c. With twice as much charge now on each inner face, the electric field at any point between the plates has the magnitude

$$E = \frac{\sigma}{\epsilon_0}.$$

# 23-6

## Applying Gauss' Law: Spherical Symmetry

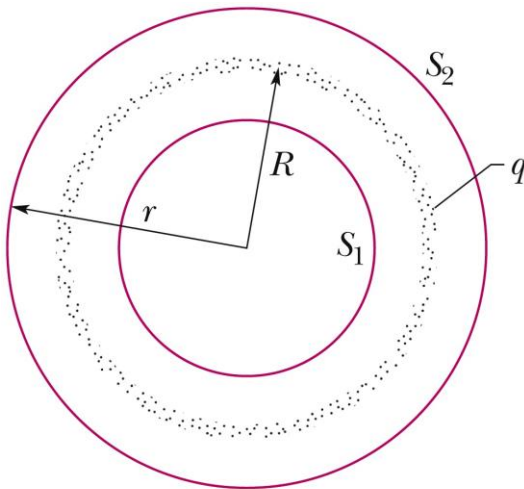
## 23-6 Applying Gauss' Law: Spherical Symmetry



A shell of uniform charge attracts or repels a charged particle that is outside the shell as if all the shell's charge were concentrated at the center of the shell.



If a charged particle is located inside a shell of uniform charge, there is no electrostatic force on the particle from the shell.



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A thin, uniformly charged, spherical shell with total charge  $q$ , in cross section. Two Gaussian surfaces  $S_1$  and  $S_2$  are also shown in cross section. Surface  $S_2$  encloses the shell, and  $S_1$  encloses only the empty interior of the shell.

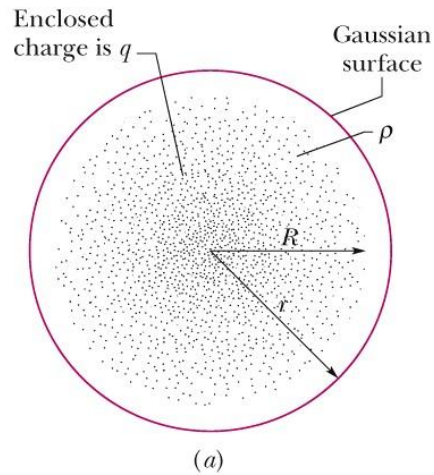
In the figure, applying Gauss' law to surface  $S_2$ , for which  $r \geq R$ , we would find that

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \quad (\text{spherical shell, field at } r \geq R).$$

And, applying Gauss' law to surface  $S_1$ , for which  $r < R$ ,

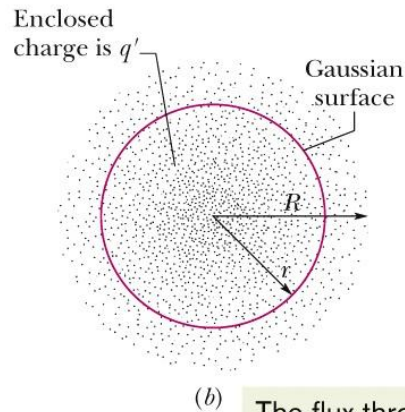
$$E = 0 \quad (\text{spherical shell, field at } r < R),$$

## 23-6 Applying Gauss' Law: Spherical Symmetry



Inside a sphere with a uniform volume charge density, the field is radial and has the magnitude

$$E = \left( \frac{q}{4\pi\epsilon_0 R^3} \right) r \quad (\text{uniform charge, field at } r \leq R).$$



The flux through the surface depends on only the *enclosed* charge.

where  $q$  is the total charge,  $R$  is the sphere's radius, and  $r$  is the radial distance from the center of the sphere to the point of measurement as shown in figure.

A concentric spherical Gaussian surface with  $r > R$  is shown in (a). A similar Gaussian surface with  $r < R$  is shown in (b).



# 23-4

## SUMMARY

# 23 Summary

## Gauss' Law

- Gauss' law is

$$\epsilon_0 \Phi = q_{\text{enc}} \quad \text{Eq. 23-6}$$

- the net flux of the electric field through the surface:

$$\Phi = \oint \vec{E} \cdot d\vec{A} \quad \text{Eq. 23-6}$$

## Applications of Gauss' Law

- surface of a charged conductor

$$E = \frac{\sigma}{\epsilon_0} \quad \text{Eq. 23-11}$$

- Within the surface  $E=0$ .
- line of charge

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \quad \text{Eq. 23-12}$$

- Infinite non-conducting sheet

$$E = \frac{\sigma}{2\epsilon_0} \quad \text{Eq. 23-13}$$

- Outside a spherical shell of charge

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \quad \text{Eq. 23-15}$$

- Inside a uniform spherical shell

$$E = 0 \quad \text{Eq. 23-16}$$

- Inside a uniform sphere of charge

$$E = \left( \frac{q}{4\pi\epsilon_0 R^3} \right) r. \quad \text{Eq. 23-20}$$