AMERICAN UNIVERSITY OF BEIRUT
Mathematics Department
Math 218 - Final Exam
Fall 2006-2007

Name:...Solution....................

ID:...

Section: 4 (@ 12:30) 5 (@ 9:30)

Time: 120 min

Directions:

- Read each question carefully before you answer.
- If you are asked to answer by TRUE or FALSE, make sure that you do not give an abrupt answer.

I- Give a ***precise*** definition of the following:

(a) **(4 points)** A vector space:

No definitions in this semester's final, but make sure that you know them, they are extremely important. You can't solve problems if you do not know what they are talking about.

(b) **(3 points)** A basis for a vector space:

(c) **(3 points)** The rank of a matrix:

(d) **(3 points)** A linear transformation between two vector spaces:

(e) **(3 points)** The nullity of a linear transformation:

II- Let

$$A = \begin{bmatrix} 1 & 4 & -3 & 0 \\ 2 & 10 & 0 & 4 \\ 3 & 13 & -7 & 3 \end{bmatrix}$$

(a) **(2.5 points)** Find a basis for the row space of A.

$\longrightarrow$

(continue your answer here)

$$\begin{bmatrix} 1 & 4 & -3 & 0 \\ 2 & 10 & 0 & 4 \\ 3 & 13 & -7 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & -3 & 0 \\ 0 & 2 & 6 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 4 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Therefore, a basis for the row space of A is

$\{(1,4,-3,0), (0,1,3,2), (0,0,1,-1)\}$

(Row vectors in the REF of A with leading 1's)

(b) **(2.5 points)** Find a basis for the column space of A.

From the REF of A above, a basis for the column space of A is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 10 \\ 13 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -7 \end{bmatrix} \right\}$

(Column vectors of A corresponding to column vectors of the REF of A that contain a leading 1)

(c) **(3 points)** Use (b) to show that the system

call this B

$$\begin{bmatrix} 1 & 4 & -3 \\ 2 & 10 & 0 \\ 3 & 13 & -7 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$$

is consistent, ***without solving it***.

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 10 \\ 13 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -7 \end{bmatrix} \right\}$ is a basis for the column space of the previous matrix A. So,

$\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$ is in $\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 10 \\ 13 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -7 \end{bmatrix} \right\} = \text{ColumnSpace}_B$

Therefore we have $Bx = b$, $b \in \text{ColumnSpace}_B$

III- Let

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) **(4 points)** Find the eigenvalues of B.

λ is an eigenvalue of B

$\iff \det(\lambda I - B) = 0$

$\iff \begin{vmatrix} \lambda-1 & -1 & -1 \\ 0 & \lambda-2 & -2 \\ 0 & 0 & \lambda-3 \end{vmatrix} = 0 \iff (\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\iff \lambda_1 = 1 \;,\; \lambda_2 = 2 \quad \lambda_3 = 3$

(c) **(8 points)** Find bases for the eigenspaces of B.

$\text{EigenSpace}_{\lambda_1} = \{ x \in R^3 \mid Av = \lambda_1 x \} = \text{NullSpace}_{\lambda_1 I - A}$

$$\left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_1 = t \\ x_3 = 0 \\ x_2 = 0 \end{array}$$

$\text{EigenSpace}_{\lambda_1} = \{ x \in R^3 \mid x = (t, 0, 0) \text{ where } t \in R \}$

Therefore, $\{(1,0,0)\}$ is a basis for $\text{EigenSpace}_{\lambda_1}$

(Continue your answer here)

EigenSpace$_{\lambda_2}$ = NullSpace$_{2I-A}$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = 0$
$x_2 = t$
$x_1 = t$

basis is $\{(1,1,0)\}$

EigenSpace$_{\lambda_3}$ = NullSpace$_{3I-A}$

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$
$x_2 = 2t$
$x_1 = \frac{3}{2}t$

basis is $\{(\frac{3}{2}, 2, 1)\}$.

(b) **(5 points)** Is B diagonalizable? If yes, find a matrix P that diagonalizes B and determine $P^{-1}BP$.

3 distinct eigenvalues so B is diagonalizable

$$P = \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}BP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

IV- Consider the system $Ax = b$.

(a) (i) **(6 points)** Find a least squares solution for the system if:

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 6 \\ 3 \\ 9 \\ 6 \end{bmatrix}$$