■ Solutions to Exercises

1. $\mathbf{u} \cdot \mathbf{v} = (0)(1) + (1)(-1) + (3)(2) = 5$

2. $\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} = \frac{0-1+6}{1+1+4} = \frac{5}{6}$

3.

$$\mathbf{u} \cdot (\mathbf{v} + 2\mathbf{w}) = \mathbf{u} \cdot \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = 0 + 1 - 12 = -11$$

4. $\frac{\mathbf{u} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w} = \frac{0+1-9}{1+1+9} \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = -\frac{8}{11} \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

5. $\| \mathbf{u} \| = \sqrt{1^2 + 5^2} = \sqrt{26}$

6. $\|\mathbf{u} - \mathbf{v}\| = \sqrt{\begin{bmatrix} -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \end{bmatrix}} = \sqrt{17}$

7. Divide each component of the vector by the norm of the vector, so that $\frac{1}{\sqrt{26}} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ is a unit vector in the direction of $\mathbf{u}$.

8. Since $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{7}{\sqrt{26}\sqrt{5}}$, then the vectors are not orthogonal.

9. $\frac{10}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

10. The vector $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$ if and only if $w_1 + 5w_2 = 0$ and $2w_1 + w_2 = 0$, that is, $w_1 = 0 = w_2$.

11. $\| \mathbf{u} \| = \sqrt{(-3)^2 + (-2)^2 + 3^2} = \sqrt{22}$

12. $\|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})}$

$$= \sqrt{\begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}} = \sqrt{41}$$

13. $\frac{1}{\sqrt{22}} \begin{bmatrix} -3 \\ -2 \\ 3 \end{bmatrix}$

14. Since $\cos \theta = -\frac{4}{11\sqrt{2}} \neq 0$, then the vectors are not orthogonal.

15. $-\frac{3}{\sqrt{11}} \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix} = \frac{3}{\sqrt{11}} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

16. A vector $\mathbf{w}$ is orthogonal to both vectors if and only if $\begin{cases} -3w_1 - 2w_2 + 3w_3 = 0 \\ -w_1 - w_2 - 3w_3 = 0 \end{cases} \Lef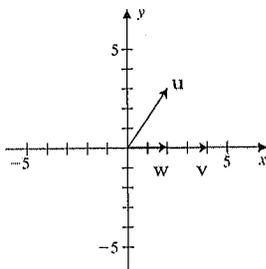trightarrow$ $w_1 = 9w_3, w_2 = -12w_3$. So all vectors in $\mathbf{span} \left\{ \begin{bmatrix} 9 \\ -12 \\ 1 \end{bmatrix} \right\}$ are orthogonal to the two vectors.

17. Since two vectors in $\mathbb{R}^2$ are orthogonal if and only if their dot product is zero, solving $\begin{bmatrix} c \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0$, gives $-c + 6 = 0$, that is, $c = 6$.

18. $\begin{bmatrix} -1 \\ c \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = 0 + 2c - 2 = 0 \Leftrightarrow c = 1$
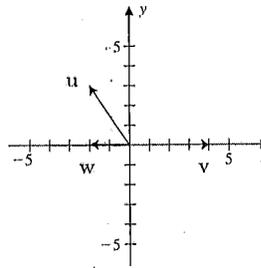
19. The pairs of vectors with dot product equaling 0 are $\mathbf{v}_1 \perp \mathbf{v}_2$, $\mathbf{v}_1 \perp \mathbf{v}_4$, $\mathbf{v}_1 \perp \mathbf{v}_5$, $\mathbf{v}_2 \perp \mathbf{v}_3$, $\mathbf{v}_3 \perp \mathbf{v}_4$, and $\mathbf{v}_3 \perp \mathbf{v}_5$.

20. The vectors $\mathbf{v}_2$ and $\mathbf{v}_5$ are in the same direction.

21. Since $\mathbf{v}_3 = -\mathbf{v}_1$, the vectors $\mathbf{v}_1$ and $\mathbf{v}_3$ are in opposite directions.

22. $\|\mathbf{v}_4\| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1$
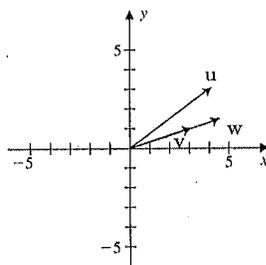
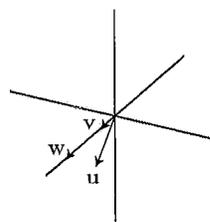**23.** $\mathbf{w} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

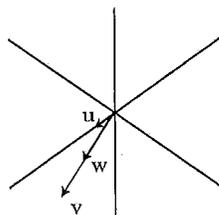**24.** $\mathbf{w} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

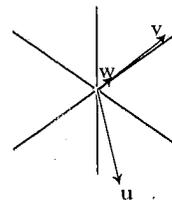**25.** $\mathbf{w} = \frac{3}{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

26. $\mathbf{w} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$

27. $\mathbf{w} = \frac{1}{6}\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$

28. $\mathbf{w} = \frac{3}{13}\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

29. Let $\mathbf{u}$ be a vector in $\text{span}\{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_n}\}$. Then there exist scalars $c_1, c_2, \cdots, c_n$ such that

$$\mathbf{u} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_n\mathbf{u_n}.$$

Using the distributive property of the dot product gives

$$\begin{aligned} \mathbf{v} \cdot \mathbf{u} &= \mathbf{v} \cdot (c_1\mathbf{u_1} + c_2\mathbf{u_2} + \ldots + c_n\mathbf{u_n}) \\ &= c_1(\mathbf{v} \cdot \mathbf{u_1}) + c_2(\mathbf{v} \cdot \mathbf{u_2}) + \cdots + c_n(\mathbf{v} \cdot \mathbf{u_n}) \\ &= c_1(0) + c_2(0) + \cdots + c_n(0) = 0. \end{aligned}$$

30. If $\mathbf{u}$ and $\mathbf{w}$ are in S and c is a scalar, then

$$(\mathbf{u} + c\mathbf{w}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} + c(\mathbf{w} \cdot \mathbf{v}) = 0 + 0 = 0$$

and hence, S is a subspace.

31. Consider the equation

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{0}.$$

We need to show that the only solution to this equation is the trivial solution $c_1 = c_2 = \cdots = c_n = 0$. Since

$$\mathbf{v_1} \cdot (c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}) = \mathbf{v_1} \cdot \mathbf{0}, \text{ we have that } c_1(\mathbf{v_1} \cdot \mathbf{v_1}) + c_2(\mathbf{v_1} \cdot \mathbf{v_2}) + \cdots + c_n(\mathbf{v_1} \cdot \mathbf{v_n}) = 0.$$

Since S is an orthogonal set of vectors, $\mathbf{v_i} \cdot \mathbf{v_j} = 0$, whenever $i \neq j$, so this last equation reduces to $c_1||\mathbf{v_1}||^2 = 0$. Now since the vectors are nonzero their lengths are positive, so $||\mathbf{v_1}|| \neq 0$ and hence, $c_1 = 0$. In a similar way we have that $c_2 = c_3 = \cdots = c_n = 0$. Hence, S is linearly independent.

32. Since $AA^{-1} = I$, then $\sum_{k=1}^{n} a_{ik}a_{kj}^{-1} = 0$ for $i \neq j$.

33. For any vector $\mathbf{w}$, the square of the norm and the dot product are related by the equation $||\mathbf{w}||^2 = \mathbf{w} \cdot \mathbf{w}$. Then applying this to the vectors $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ gives

$$\begin{aligned}
||\mathbf{u} + \mathbf{v}||^2 + ||\mathbf{u} - \mathbf{v}||^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) + (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\
&= \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\
&= 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2.
\end{aligned}$$

34. a. The normal vector to the plane, $\mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, is orthogonal to every vector in the plane.

b. $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

35. If the column vectors of A form an orthogonal set, then the row vectors of A^t are orthogonal to the column vectors of A. Consequently, $(A^t A)_{ij} = 0$ if $i \neq j$. If $i = j$, then $(A^t A)_{ii} = ||\mathbf{A_i}||^2$. Thus,

$$A^t A = \begin{bmatrix} ||\mathbf{A_1}||^2 & 0 & \cdots & 0 \\ 0 & ||\mathbf{A_2}||^2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & ||\mathbf{A_n}||^2 \end{bmatrix}.$$

36. First notice that $\mathbf{u} \cdot (A\mathbf{v}) = \mathbf{u}^t(A\mathbf{v})$. So

$$\mathbf{u} \cdot (A\mathbf{v}) = \mathbf{u}^t(A\mathbf{v}) = (\mathbf{u}^t A)\mathbf{v} = (A^t\mathbf{u})^t\mathbf{v} = (A^t\mathbf{u}) \cdot \mathbf{v}.$$

37. Suppose that $(A\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (A\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. By Exercise 36, $\mathbf{u} \cdot (A\mathbf{v}) = (A^t\mathbf{u}) \cdot \mathbf{v}$. Thus,

$$(A^t\mathbf{u}) \cdot \mathbf{v} = (A\mathbf{u}) \cdot \mathbf{v}$$

for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Let $\mathbf{u} = \mathbf{e_i}$ and $\mathbf{v} = \mathbf{e_j}$, so $(A^t)_{ij} = A_{ij}$. Hence $A^t = A$, so A is symmetric. For the converse, suppose that $A = A^t$. Then by Exercise 36,

$$\mathbf{u} \cdot (A\mathbf{v}) = (A^t\mathbf{u}) \cdot \mathbf{v} = (A\mathbf{u}) \cdot \mathbf{v}.$$

Exercise Set 6.2

Inner products are generalizations of the dot product on the Euclidean spaces. An inner product on the vector space V is a mapping from $V \times V$, that is the input is a pair of vectors, to $\mathbb{R}$, so the output is a number. An inner product must satisfy all the same properties of the dot product discussed in Section 6.1. So to