AMERICAN UNIVERSITY OF BEIRUT
MATHEMATICS 218
FALL SEMESTER 2009-2010
FINAL EXAMINATION
VERSION A

Time: 120 minutes　　　　Date: January 27, 2010

Name:..

ID:...

Circle your section number in the table below:

Instructors	Azar	Egeileh	El Khoury	Fuleihan	Karam	Nassif
Section	8	5	1	6	4	2
Section			3		9	7

QUESTION	GRADE
1	/25
2	/25
3	/25
4	/26
5	/30
6	/34
7	/35
TOTAL GRADE	/200

Answer the following seven sets of questions; the back of pages may be used as scratch.

1. Let $T : R^3 \to R^4$ be defined by:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y + 3z \\ x + 2y \\ y + z \\ z \end{bmatrix}$$

(a) Show that T is a linear transformation. (8 points)
(b) Show that T is one-to-one. (6 points)
(c) What is the rank of the matrix representing T. (4 points)
(d) Find a basis for the range of T. (7 points)

2. Let $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 3 \\ 3 & 5 & -6 & -5 \end{pmatrix}$

(a) Find a basis for the $N(A)$. (10 points)

(b) Determine a basis for $N(A)^{\perp}$ with respect to the Euclidean inner product. (7 points)

(c) What is the dimension of $N(A^T)$? (8 points)

3. Let $A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$

(a) Find the eigenvalues of A. (10 points)

(b) Find P that diagonalizes A. (10 points)

(c) Deduce $P^{-1}AP$. (5 points)

4. (*a*) Without solving, find condition(s) on α and β for the following system to have exactly one solution: (10 points)

$$\begin{cases} x+y+\alpha z=1 \\ x+y+\beta z=2 \\ \alpha x+\beta y+z=-1 \end{cases}$$

(*b*) For $\alpha=\beta=0$ find the least squares solution(s) of the linear system. (16 points)

5. Let $W = \{p(x) \in P_3 / \; xp(2) - p(1) = 0\}$

(a) Prove that W is a subspace of P_3. (8 points)

(b) Find a basis for W. (12 points)

(c) Let $T : P_3 \longrightarrow P_1$ be the linear transformation defined by:

$$T(p(x)) = xp(2) - p(1)$$

Show that T is onto. (10 points)

6. Let P_2 be the set of polynomials of degree less or equal than 2. Define the inner product on P_2 by:

$$< p, q >= a_0 b_0 + a_1 b_1 + a_2 b_2$$

where $p(x) = a_0 + a_1 x + a_2 x^2$ and $q(x) = b_0 + b_1 x + b_2 x^2$.

Let $p(x) = 1 + x$, $q(x) = 1 + 2x^2$ and $W = \text{span}\,\{p, q\}$

(a) Find λ such that $(q - \lambda p)$ and p are orthogonal. (6 points)
(b) Show that $\{p, q\}$ is a basis for W. (4 points)
(c) Use the Gram-Schmidt process to find an orthogonal basis for W. (10 points)
(d) Find a basis for $W^{\perp}$. (10 points)
(e) Deduce an orthogonal basis for P_2. (4 points)

7. Prove the following statements:

(*a*) $T : V \rightarrow V$ such that T is one-to-one, prove that if $\{v_1, \ldots, v_n\}$ are linearly independent in V, then $\{T(v_1), \ldots T(v_n)\}$ are linearly independent.

(10 points)

(*b*) Let V be an inner product space. Show that:

$$||u|| = ||v|| <=> u + v \text{ and } u - v \text{ are orthogonal}$$

(10 points)

(c) Let B be an $n \times n$ invertible matrix and $T : M_{n\times n} \longrightarrow M_{n\times n}$ be defined by $T(A) = AB$. Show that T is an isomorphism. (15 points)