MATHEMATICS 218
SPRING SEMESTER 2008-09
FINAL EXAMINATION

Time: 120 minutes　　　　Date: June 3, 2009

Name:——————————————

ID Number:——————————————

Circle your section number in the table below:

Instructors	Ejeili	Fuleihan	Itani	Karam	El Khoury	Lyzzaik	Nassif	Tannous
Section	7	3	11	1	10	6	2	4
Section		5		12		8	9	

QUESTION	GRADE
1	/25
2	/20
3	/20
4	/20
5	/20
6	/20
7	/20
8	/25
9	/30
TOTAL GRADE	/200

Answer the following nine sets of questions on the allocated pages; the back of pages may be used if needed

1. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ the transformation defined by

$$T(x, y, z, w) = (x + y - z + w, 2x + y + 4z + w, 3x + y + 9z).$$

(a) Show that T is a linear transformation. (4 points)
(b) Find the standard matrix $[T]$. (5 points)
(c) Find bases for the kernel of and range of T. (12 points)
(d) Find the rank and nullity of T. (4 points)

2. Let $T : P_2 \to \mathbb{R}^3$ be the function defined by the formula

$$T(\mathbf{p}(x)) = \begin{bmatrix} \mathbf{p}(1) \\ \mathbf{p}(2) \\ \mathbf{p}(3) \end{bmatrix};$$

here P_2 is the vector space of all real polynomials of degree at most 2.

(a) Show that T is a linear transformation. (8 points)

(b) Show that T is one-to-one. (8 points)

(c) Show that T is onto. (4 points)

3. Let P_3 be the set of all polynomials of degree at most 3, and let

$$W = \{ax^3 + bx^2 + cx + d : b + c + d = 0\}.$$

(a) Show that W is a subspace of P_3. (6 points)
(b) Find a basis S for W. (6 points)
(c) Give one vector in P_3 but not in W. (2 points)
(d) Complete S to a basis for P_3. (6 points)

4. Let

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 0 & 4 & 2 \\ 3 & 2 & 8 & 7 \end{bmatrix}$$

(a) Find a basis for the row space of A and a basis for its orthogonal complement. (16 points)

(b) Find a subset of the column vectors of A that forms a basis for the column space of A. (4 points)

5. Let P_2 be the set of all polynomials of degree at most 2, and let $\mathbf{p}(x) = a_0 + a_1x + a_2x^2$ and $\mathbf{q}(x) = b_0 + b_1x + b_2x^2$ be in P_2. Define on P_2 the operation

$$< \mathbf{p}, \mathbf{q} >= a_0b_0 + 2a_1b_1 + 3a_2b_2.$$

(a) Show that $< .,. >$ is an inner product on P_2. (10 points)

(b) Show that

$$(a_0b_0 + 2a_1b_1 + 3a_2b_2)^2 \leq (a_0^2 + 2a_1^2 + 3a_2^2)(b_0^2 + 2b_1^2 + 3b_2^2).$$

(5 points)

(c) Determine the cosine of the angle between the polynomials

$$1 + x - x^2 \quad \text{and} \quad 1 - x + x^2.$$

(5 points)

6. Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}.$$

(a) Use the Gram-Schmidt process to transform the column vectors of A to an orthonormal basis of $\mathbb{R}^3$. (14 points)

(b) Find the QR-decomposition of A. (6 points)

7. Let

$$A = \begin{bmatrix} -2 & 3 \\ 1 & -2 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

(a) Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$.

(16 points)

(b) Find the orthogonal projection of $\mathbf{b}$ on the column space of A.

(4 points)

8. Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(a) Find the eigenvalues of A. (5 points)

(b) Show that A is diagonalizable. (10 points)

(c) Find a matrix P that diagonalizes A and determine $P^{-1}AP$. (5 points)

(d) Determine A^{10}. (5 points)

9. Indicate whether each of the following statements is TRUE (T) or FALSE (F) without justifying your answer. (3 points each)

—-(a) If A is an $n \times n$ matrix that satisfies $AA^T = I$, then $\det(A) = 1$.

—-(b) If $A^2 = A$ and λ is an eigenvalue of A, then $\lambda = 0$ or $\lambda = 1$.

—-(c) If A is an $n \times n$ matrix invertible matrix, then the orthogonal complement of its nullspace is $\mathbb{R}^n$.

—-(d) A square matrix is diagonalizable if and only if $\lambda = 0$ is an eigenvalue.

—-(e) Any linear system $A\mathbf{x} = \mathbf{b}$ satisfies $\text{rank}[A|\mathbf{b}] = \text{rank}(A)$.

—-(f) Any matrix A can be expressed as a product of elementary matrices.

—-(g) If dim V < dim W < ∞, then there exists a one-to-one linear transformation $T : V \to W$.

—-(h) If a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}$ satisfies $T(2, -1) \neq 0$, then it is onto.

—-(i) The dimension of the vector space of 3×3 matrices is 10.

—-(j) If A is an $m \times n$ matrix, then $A^T A$ is invertible if and only if the set of column vectors of A is linearly independent.