

Mathematics Department
MATH 218 - QUIZ I
Spring 2007-2008

Name:... Solution.....

ID:.....

Section: 9 (@ 9:30) 2 (@ 11:00)

Time: 60 min

I- a) (15 points) Solve the following system:

$$\left\{ \begin{array}{l} 2x_1 + 16x_2 + 6x_3 - 2x_4 + 2x_5 = 10 \\ 3x_1 + 10x_2 + 9x_3 + 5x_4 + 2x_5 = 20 \\ -8x_2 + 8x_4 + 2x_5 = 2 \end{array} \right.$$

$$\left[\begin{array}{ccccc|c} 2 & 16 & 6 & -2 & 2 & 10 \\ 3 & 10 & 9 & 5 & 2 & 20 \\ 0 & -8 & 0 & 8 & 2 & 2 \end{array} \right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccccc|c} 2 & 16 & 6 & -2 & 2 & 10 \\ 1 & -6 & 3 & 7 & 0 & 10 \\ 0 & -8 & 0 & 8 & 2 & 2 \end{array} \right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccccc|c} 1 & -6 & 3 & 7 & 0 & 10 \\ 2 & 16 & 6 & -2 & 2 & 10 \\ 0 & -8 & 0 & 8 & 2 & 2 \end{array} \right] \xrightarrow{2R_3+R_2} \left[\begin{array}{ccccc|c} 1 & -6 & 3 & 7 & 0 & 10 \\ 2 & 0 & 6 & 14 & 6 & 14 \\ 0 & -8 & 0 & 8 & 2 & 2 \end{array} \right]$$

$$\xrightarrow{-2R_1+R_2} \left[\begin{array}{ccccc|c} 1 & -6 & 3 & 7 & 0 & 10 \\ 0 & 12 & 0 & 0 & 6 & -6 \\ 0 & -8 & 0 & 8 & 2 & 2 \end{array} \right] \xrightarrow{\frac{1}{6}R_2} \left[\begin{array}{ccccc|c} 1 & -6 & 3 & 7 & 0 & 10 \\ 0 & 2 & 0 & 0 & 1 & -1 \\ 0 & -8 & 0 & 8 & 2 & 2 \end{array} \right]$$

$$\xrightarrow{\frac{4}{3}R_2+R_1} \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 7 & 3 & 7 \\ 0 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 8 & 6 & -2 \end{array} \right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 7 & 3 & 7 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} \end{array} \right]$$

$$\xrightarrow{-7R_3+R_1} \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & -\frac{9}{4} & \frac{35}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} \end{array} \right]$$

Let $x_3 = s$
 $x_5 = t$
 $x_1 = \frac{35}{4} + \frac{9}{4}t - 3s$

$$x_2 = -\frac{1}{2} - \frac{1}{2}t$$

$$x_4 = -\frac{1}{4} - \frac{3}{4}t$$

$$s, t \in \mathbb{R}$$

-1 calculation
5 wrong allocation
of variables

b) Consider the matrix $A = \begin{bmatrix} 5 & 1 & 2 \\ 8 & 1 & 3 \end{bmatrix}$

(i) (9 points) Let R be the reduced row echelon form of A . Find elementary matrices E, F, G and H such that $R = EFGHA$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 2 \\ 8 & 1 & 3 \end{bmatrix} \xrightarrow{-5R_1+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 8 & 1 & 3 \end{bmatrix} \xrightarrow{-8R_1+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_3+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-5R_1+R_2} \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = H$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-8R_1+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -8 & 0 & 1 \end{bmatrix} = G \quad 4$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = F$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_3+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = E$$

2 pts for
the rule

(ii) (9 points) Deduce whether A is invertible or not. Then, if A is invertible, write A^{-1} as a product of elementary matrices and justify your answer.

5 RREF of A is I , hence by the equivalence theorem A is invertible.

$$I = EFGHA \quad (\text{part (i)})$$

$$4 IA^{-1} = EFGHAA^{-1}$$

$$A^{-1} = EFGH.$$

Other justifications are possible

reasons for your solution.

- Given an $n \times n$ matrix A , if $\det(A) \neq 0$ then A is invertible.
- Given two $n \times n$ matrices A and B , AB is invertible $\iff A$ is invertible and B is invertible.
- Given an $n \times n$ matrix A , the reduced row echelon form of A and A are either both invertible or both not invertible.

solved in class

-3 rules of mathematical logic
-7 for taking a special case

$$\text{Let } A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \quad A^3 = \begin{bmatrix} a^3 & 0 & 0 \\ 0 & b^3 & 0 \\ 0 & 0 & c^3 \end{bmatrix}$$

$$A = A^3 \Leftrightarrow \begin{cases} a = a^3 \\ b = b^3 \\ c = c^3 \end{cases} \quad \begin{cases} a^3 - a = 0 \\ b^3 - b = 0 \\ c^3 - c = 0 \end{cases} \quad \begin{cases} a(a^2 - 1) = 0 \\ b(b^2 - 1) = 0 \\ c(c^2 - 1) = 0 \end{cases}$$

Therefore $\frac{a}{b} \in \{0, 1, -1\}$

The set of diagonal 3×3 matrices where $A = A^3$ is the set of matrices $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ where $a, b, c \in \{0, 1, -1\}$

BONUS: (3.5 points) How many diagonal $n \times n$ matrices A are there such that $A = A^4$?

$$a_{ii}^4 = a_{ii} \quad a_{ii}(a_{ii}^3 - 1) = 0 \quad a_{ii} \in \{0, 1\}$$

two options for each entry on the main diagonal hence we have 2^n matrices.

(b) (9 points) Without directly computing the determinant, show that,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \frac{-1}{2} \begin{vmatrix} a_1 & c_1^2 + b_1 & -2c_1 \\ a_2 & c_1c_2 + b_2 & -2c_2 \\ a_3 & c_1c_3 + b_3 & -2c_3 \end{vmatrix}.$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1^2 + b_1 & c_1 \\ a_2 & c_1c_2 + b_2 & c_2 \\ a_3 & c_1c_3 + b_3 & c_3 \end{vmatrix}$$

$c_1 C_3 + C_2$

$$= -\frac{1}{2} \begin{vmatrix} a_1 & c_1^2 + b_1 & -2c_1 \\ a_2 & c_1c_2 + b_2 & -2c_2 \\ a_3 & c_1c_3 + b_3 & -2c_3 \end{vmatrix} \quad \text{with } -2C_3$$

$$(A - I)^{-1} = I + 2A + A^2$$

$$(A - I)(I + 2A + A^2) \stackrel{?}{=} I$$

$$\begin{aligned}(A - I)(I + 2A + A^2) &= A + 2A^2 + A^3 - I - 2A - A^2 \\ &= A^3 + A^2 - A - I\end{aligned}$$

$$\text{But } A^3 + A^2 - A - 2I = 0$$

$$A^3 + A^2 - A - I = I$$

$$\text{therefore } (A - I)(I + 2A + A^2) = I$$

$$\text{Hence by a theorem, } (A - I)^{-1} = I + 2A + A^2$$

(b) (7 points) Assuming that the stated inverses exist, show that if $B^{-1}(B^{-1}C)^T$ is symmetric, then the inverse of $(C^T B^T + C)^{-1}B$ is $B^{-1}C(B^T + I)$.

$$(B^{-1}(B^{-1}C)^T)^T = B^{-1}(B^{-1}C)^T$$

$$B^{-1}C(B^{-1})^T = B^{-1}C^T(B^{-1})^T$$

Multiply by B^T from the right, since $(B^{-1})^T = (B^T)^{-1}$ we get,

$$B^{-1}C = B^{-1}C^T$$

Multiply by B from the left,

$$\boxed{C = C^T}$$

3.5

$$((C^T B^T + C)^{-1} B)^{-1} = B^{-1}(C^T B^T + C)$$

$$= B^{-1}(C^T B^T + C^T)$$

$$= B^{-1}C^T(B^T + I)$$

3.5

give a short logical argument or a counterexample.

- 1- There is no square matrix A such that $\det(AA^T) = -2$. True

Suppose there is a square matrix A such that
 $\det(AA^T) = -2$
 $\det(A) \det(A^T) = -2$, $\det(A) \det(A) = -2$
 $(\det(A))^2 = -2$ impossible

- 2- If A is an invertible matrix and $AB = AC$ then $B = C$. True

$$\begin{aligned} AB &= AC \\ A^{-1}AB &= A^{-1}AC \\ B &= C \end{aligned}$$

- 3- A linear system with more unknowns than equations has infinitely many solutions. False

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 3 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

the linear system represented by this augmented matrix has 6 unknowns and 3 equations but has no solution.

- 4- If $Ax = b$ has a unique solution for all b then A can be written as a product of elementary matrices. True

By the equivalence theorem

- 5- If $A + B$ is symmetric, then both A and B are symmetric. False

If $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 7 & 2 \\ 3 & 3 \end{bmatrix}$ both are not symmetric

$$A+B = \begin{bmatrix} 8 & 6 \\ 6 & 5 \end{bmatrix} \text{ symmetric}$$

GOOD LUCK