- The matrix can have three distinct real eigenvalues and three linearly independent eigenvectors.
- The matrix can have two distinct real eigenvalues such that one has algebraic multiplicity 2 and the other algebraic multiplicity 1. But the matrix can still have three linearly independent eigenvectors, two from the eigenvalue of multiplicity 2 and one form the other.

Solutions to Exercises

1. Since the matrix equation $\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is satisfied if and only if $\lambda = 3$, the eigenvalue corresponding to the eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $\lambda = 3$.

2. Since the matrix equation $\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is satisfied if and only if $\lambda = -2$, the eigenvalue corresponding to the eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is $\lambda = -2$.

3. The corresponding eigenvalue is $\lambda = 0$.

4. The corresponding eigenvalue is $\lambda = -2$.

5. The corresponding eigenvalue is $\lambda = 1$.

6. The corresponding eigenvalue is $\lambda = -1$.

7. a. The characteristic equation is $\det(A - \lambda I) = 0$, that is,

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 2 \\ 3 & -3-\lambda \end{vmatrix} = (-2-\lambda)(-3-\lambda) - 6 = \lambda^2 + 5\lambda = 0.$$

b. Since the eigenvalues are the solutions to the characteristic equation $\lambda^2 + 5\lambda = \lambda(\lambda + 5) = 0$, the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -5$. **c.** The corresponding eigenvectors are found by solving, respectively, $\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}\mathbf{v} = 0$ and $\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}\mathbf{v} = -5\mathbf{v}$. Hence the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, respectively.

d.

$$A\mathbf{v}_1 = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } A\mathbf{v}_2 = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ -15 \end{bmatrix} = (-5)\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

8. a. $\lambda^2 + 4\lambda + 3 = 0$ **b.** $\lambda_1 = -1, \lambda_2 = -3$ **c.** $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **d.** $\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-3)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

9. a. $(\lambda - 1)^2 = 0$ **b.** $\lambda_1 = 1$ **c.** $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

10. a. $\lambda^2 + 3\lambda + 2 = 0$ **b.** $\lambda_1 = -1, \lambda_2 = -2$ **c.** $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-2)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

11. a. The characteristic equation $\det(A - \lambda I) = 0$, is

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{vmatrix} = 0.$$

Expanding down column one, we have that

$$0 = \begin{vmatrix} -1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{vmatrix} = (-1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} = (1+\lambda)^2(1-\lambda).$$

b. $\lambda_1 = -1$, $\lambda_2 = 1$ **c.** $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

d.

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

12. a. $\lambda(\lambda+1)(\lambda-1) = 0$ **b.** $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = 1$ **c.** $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

13. a. $(\lambda-2)(\lambda-1)^2 = 0$ **b.** $\lambda_1 = 2$, $\lambda_2 = 1$ **c.** $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = (2)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = (1)\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

14. a. $(\lambda-1)^3 = 0$ **b.** $\lambda = 1$ **c.** $\mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

15. a. $(\lambda+1)(\lambda-2)(\lambda+2)(\lambda-4) = 0$ **b.** $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = -2$, $\lambda_4 = 4$

c. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ **d.** For the eigenvalue $\lambda = -1$, the verification is

$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. The other cases are similar.

16. a. $(\lambda+1)(\lambda-1)(\lambda-2)(\lambda-3) = 0$ **b.** $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_4 = 3$ **c.** $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$,

$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -7 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$