37. Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$. Then a line L through the origin in the direction of the vector $\mathbf{v}$ is given by all scalar multiples of the vector $\mathbf{v}$. That is, $L = \{t\mathbf{v} | \ t \in \mathbb{R}\}$. Now, let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be an isomorphism. Since T is linear, then $T(t\mathbf{v}) = tT(\mathbf{v})$. Also, by Theorem 8, $T(\mathbf{v})$ is nonzero. Hence, the set $L' = \{tT(\mathbf{v}) | \ t \in \mathbb{R}\}$ is also a line in $\mathbb{R}^3$ through the origin. A plane P is given by the span of two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$. That is, $P = \{s\mathbf{u} + t\mathbf{v} | \ s, t \in \mathbb{R}\}$. Then $T(s\mathbf{u}+t\mathbf{v}) = sT(\mathbf{u})+tT(\mathbf{v})$, and since T is an isomorphism $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly independent and hence, $P' = T(P) = \{sT(\mathbf{u})+tT(\mathbf{v}) | \ s, t \in \mathbb{R}\}$ is a plane.

Exercise Set 4.4

If A is an $m \times n$ matrix, a mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by the matrix product $T(\mathbf{v}) = A\mathbf{v}$ is a linear transformation. In Section 4.4, it is shown how every linear transformation $T : V \longrightarrow W$ can be described by a matrix product. The matrix representation is given relative to bases for the vector spaces V and W and is defined using coordinates relative to these bases. If $B = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is a basis for V and B' a basis for W, two results are essential in solving the exercises:

- The matrix representation of T relative to B and B' is defined by

$$[T]_B^{B'} = [\ [T(\mathbf{v_1})]_{B'} \ [T(\mathbf{v_2})]_{B'} \ \ldots \ [T(\mathbf{v_n})]_{B'} \ ].$$

- Coordinates of $T(\mathbf{v})$ can be found using the formula

$$[T(\mathbf{v}))]_{B'} = [T]_B^{B'}[\mathbf{v}]_B.$$

To outline the steps required in finding and using a matrix representation of a linear transformation define $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \\ z \end{bmatrix}$ and let

$$B = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\} \quad \text{and} \quad B' = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right\}$$

two bases for $\mathbb{R}^3$.

- Apply T to each basis vector in B.

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Find the coordinates of each of the vectors found in the first step relative to B'. Since

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1 & 1/2 \\ 1 & 0 & 1 & -1/2 & 1 & 1/2 \\ 1 & 1 & 0 & -1/2 & 0 & -1/2 \end{array}\right],$$

then

$$\left[\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right]_{B'} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \left[\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right]_{B'} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]_{B'} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

- The column vectors of the matrix representation relative to B and B' are the coordinate vectors found in the previous step.

$$[T]_B^{B'} = \begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 1 & 1/2 \\ -1/2 & 0 & -1/2 \end{bmatrix}$$

- The coordinates of any vector $T(\mathbf{v})$ can be found using the matrix product

$$[T(\mathbf{v})]_{B'} = [T]_B^{B'}[\mathbf{v}]_B.$$

- As an example, let $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}$, then after applying the operator T the coordinates relative to B' is given by

$$\left[T\left(\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}\right)\right]_{B'} = \begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 1 & 1/2 \\ -1/2 & 0 & -1/2 \end{bmatrix}\left[\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}\right]_B.$$

Since B is the standard basis the coordinates of a vector are just the components, so

$$\left[T\left(\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}\right)\right]_{B'} = \begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 1 & 1/2 \\ -1/2 & 0 & -1/2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -9/2 \\ 3/2 \end{bmatrix}.$$

This vector is not $T(\mathbf{v})$, but the coordinates relative to the basis B'. Then

$$T\left(\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{9}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -7/2 \end{bmatrix}.$$

Other useful formulas that involve combinations of linear transformations and the matrix representation are:

- $[S+T]_B^{B'} = [S]_B^{B'} + [T]_B^{B'}$ • $[kT]_B^{B'} = k[T]_B^{B'}$ • $[S \circ T]_B^{B'} = [S]_B^{B'}[T]_B^{B'}$ • $[T^n]_B = ([T]_B)^n$ • $[T^{-1}]_B = ([T]_B)^{-1}$

Solutions to Exercises

1. a. Let $B = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis. To find the matrix representation for A relative to B, the column vectors are the coordinates of $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ relative to B. Recall the coordinates of a vector relative to the standard basis are just the components of the vector. Hence, $[T]_B = [\,[T(\mathbf{e}_1]_B\ [T(\mathbf{e}_2]_B\,] = \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}$.
b. The direct computation is $T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$ and using part (a), the result is
$T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$.

2. a. $[T]_B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **b.** The direct computation is $T\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and using part (a), the result is
$T\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

3. a. Let $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis. Then $[T]_B = [\,[T(\mathbf{e}_1]_B\ [T(\mathbf{e}_2]_B\ [T(\mathbf{e}_2]_B\,] = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}$. **b.** The direct computation is $T\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix}$, and using part (a) the result is
$T\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix}$.

4. a. $[T]_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ **b.** The direct computation is $T\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$, and using part (a) the result is $T\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$.

5. a. The column vectors of the matrix representation relative to B and B' are the coordinates relative to B' of the images of the vectors in B by T. That is, $[T]_B^{B'} = \left[\left[T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right]_{B'} \left[T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)\right]_{B'} \right]$. Since B' is the standard basis, the coordinates are the components of the vectors $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$, so $[T]_B^{B'} = \begin{bmatrix} -3 & -2 \\ 3 & 6 \end{bmatrix}$.

b. The direct computation is $T\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$ and using part (a)

$$T\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} -1 \\ -2 \end{bmatrix}_B = \begin{bmatrix} -3 & -2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}.$$

6. a. $[T]_B^{B'} = \begin{bmatrix} -3 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = T\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = [T]_B^{B'}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}_B = [T]_B^{B'}\begin{bmatrix} -3/2 \\ -3 \\ 5/2 \end{bmatrix}$

7. a. The matrix representation is given by

$$[T]_B^{B'} = \left[\left[T\left(\begin{bmatrix} -1 \\ -2 \end{bmatrix}\right)\right]_{B'} \left[T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right]_{B'} \right] = \left[\left[T\left(\begin{bmatrix} -2 \\ -3 \end{bmatrix}\right)\right]_{B'} \left[T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right)\right]_{B'} \right].$$

We can find the coordinates of both vectors by considering

$$\left[\begin{array}{cc|cc} 3 & 0 & -2 & 2 \\ -2 & -2 & -3 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & 1 & \frac{13}{6} & -\frac{5}{3} \end{array}\right], \text{ so } [T]_B^{B'} = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{13}{6} & -\frac{5}{3} \end{bmatrix}.$$

b. The direct computation is $T\begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$. Using part (a) we can now find the coordinates of the image of a vector using the formula $[T(\mathbf{v})]_{B'} = [T]_B^{B'}[\mathbf{v}]_B$. and then use these coordinates to fine $T(\mathbf{v}$. That is,

$$\left[T\begin{bmatrix} -1 \\ -3 \end{bmatrix}\right]_{B'} = [T]_B^{B'}\begin{bmatrix} -1 \\ -3 \end{bmatrix}_B = [T]_B^{B'}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{8}{3} \end{bmatrix}, \text{ so } T\begin{bmatrix} -1 \\ -3 \end{bmatrix} = -\frac{2}{3}\begin{bmatrix} 3 \\ -2 \end{bmatrix} + \frac{8}{3}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}.$$

8. a. $[T]_B^{B'} = \begin{bmatrix} -1 & 1 & 1 \\ -3 & 1 & -1 \\ -3 & 1 & -2 \end{bmatrix}$ **b.** The direct computation gives $T\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}$. Using the matrix in part (a) gives $\left[T\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}\right]_{B'} = [T]_B^{B'}\left[T\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}\right]_B = [T]_B^{B'}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}$, so that

$$T\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 4\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}.$$

9. a. Since B' is the standard basis for $\mathcal{P}_2$, then $[T]_B^{B'} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. **b.** The direct computation is $T(x^2-3x+3) = x^2-3x+3$. To find the coordinates of the image, we have from part (a) that

$$\left[T(x^2-3x+3)\right]_{B'} = [T]_B^{B'}[x^2-3x+3]_B = [T]_B^{B'}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, \text{ so } T(x^2-3x+3) = 3-3x+x^2.$$

10. a. $[T]_B^{B'} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 0 & 1 \\ -3 & 1 & 3 \end{bmatrix}$ **b.** The direct computation gives $T(1-x) = \frac{d}{dx}(1-x)+(1-x) = -x$. Using the matrix in part (a) gives $[T(1-x)]_{B'} = [T]_B^{B'}[1-x]_B = [T]_B^{B'}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, so $T(1-x) = 0(-1+x)+0(-1+x+x^2)-x = -x$.

11. First notice that if $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, then $T(A) = \begin{bmatrix} 0 & -2b \\ 2c & 0 \end{bmatrix}$.

a. $[T]_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **b.** The direct computation is $T\left(\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}\right) = \begin{bmatrix} 0 & -2 \\ 6 & 0 \end{bmatrix}$. Using part (a)

$$\left[T\left(\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}\right)\right]_B = [T]_B\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 6 \end{bmatrix}$$

so

$$T\left(\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}\right) = 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 6 & 0 \end{bmatrix}.$$

12. First notice that $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 3a & b+2c \\ 2b+c & 3d \end{bmatrix}$. **a.** $[T]_B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

b. The direct computation gives $T\left(\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 5 & 6 \end{bmatrix}$. Using the matrix in part (a) gives

$$\left[\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\right]_B = [T]_B\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}_B = [T]_B\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 6 \end{bmatrix}, \text{ so } T\left(\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 5 & 6 \end{bmatrix}.$$

13. a. $[T]_B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ **b.** $[T]_{B'} = \frac{1}{9}\begin{bmatrix} 1 & 22 \\ 11 & -1 \end{bmatrix}$ **c.** $[T]_B^{B'} = \frac{1}{9}\begin{bmatrix} 5 & -2 \\ 1 & 5 \end{bmatrix}$
d. $[T]_{B'}^{B} = \frac{1}{3}\begin{bmatrix} 5 & 2 \\ -1 & 5 \end{bmatrix}$ **e.** $[T]_C^{B'} = \frac{1}{9}\begin{bmatrix} -2 & 5 \\ 5 & 1 \end{bmatrix}$ **f.** $[T]_{C'}^{B'} = \frac{1}{9}\begin{bmatrix} 22 & 1 \\ -1 & 11 \end{bmatrix}$

14. a. $[T]_B^{B''} = \begin{bmatrix} -1 & -4 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$ **b.** $[T]_{B'}^{B''} = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$ **c.** $[T]_C^{B''} = \begin{bmatrix} -4 & -1 \\ 3 & 1 \\ 2 & 1 \end{bmatrix}$ **d.** $[T]_{C'}^{B''} = \frac{1}{2}\begin{bmatrix} -3 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$
e. $[T]_B^{C''} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ -1 & -4 \end{bmatrix}$

15. **a.** $[T]_B^{B'} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ **b.** $[T]_C^{B'} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix}$ **c.** $[T]_C^{C'} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1/2 & 0 \end{bmatrix}$ **d.** $[S]_{B'}^{B} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

e. $[S]_{B'}^{B}[T]_B^{B'} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $[T]_B^{B'}[S]_{B'}^{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **f.** The function $S \circ T$ is the identity map, that is, $(S \circ T)(ax + b) = ax + b$ so S reverses the action of T.

16. **a.** $[T]_B^{B'} = \frac{1}{4}\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & -2 & 2 & -2 \\ -3 & -1 & 1 & 2 \\ -3 & 1 & -1 & 0 \end{bmatrix}$ **b.** $[T]_{B'}^{B} = \begin{bmatrix} 2 & 0 & -2 & -2 \\ 0 & 0 & -2 & 2 \\ -1 & 0 & -1 & -1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

c. $[T]_{B'}^{B'} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ -2 & 0 & -6 & 2 \\ -1 & 0 & 3 & 7 \\ -3 & 0 & 5 & 1 \end{bmatrix}$ **d.** $[I]_B^{B'} = \frac{1}{4}\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$, $[I]_{B'}^{B} = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

17. $[T]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The transformation T reflects a vector across the x-axis.

18. The transformation rotates a vector by θ radians in the counterclockwise direction.

19. $[T]_B = cI$

20. Since $T(A) = A - A^t = \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}$, then $[T]_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

21. $[T]_B^{B'} = [1\ 0\ 0\ 1]$

22. **a.** $[-3S]_B = -3[S]_B = -3\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$

23. **a.** $[2T + S]_B = 2[T]_B + [S]_B = \begin{bmatrix} 5 & 2 \\ -1 & 7 \end{bmatrix}$ **b.** $\begin{bmatrix} -4 \\ 23 \end{bmatrix}$

24. **a.** $[T \circ S]_B = [T]_B[S]_B = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}$ **b.** $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$

25. **a.** $[S \circ T]_B = [S]_B[T]_B = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ **b.** $\begin{bmatrix} -1 \\ 10 \end{bmatrix}$

26. **a.** $[2T]_B = 2[T]_B = \begin{bmatrix} 2 & -2 & -2 \\ 0 & 4 & 4 \\ -2 & 2 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} -10 \\ 16 \\ 10 \end{bmatrix}$

27. **a.** $[-3T + 2S]_B = \begin{bmatrix} 3 & 3 & 1 \\ 2 & -6 & -6 \\ 3 & -3 & -1 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 \\ -26 \\ -9 \end{bmatrix}$

28. **a.** $[T \circ S]_B = [T]_B[S]_B = \begin{bmatrix} 2 & 0 & -2 \\ 2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} -8 \\ 4 \\ 8 \end{bmatrix}$

29. **a.** $[S \circ T]_B = \begin{bmatrix} 4 & -4 & -4 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} -20 \\ -5 \\ 5 \end{bmatrix}$

30. $[T]_B = \begin{bmatrix} 4 & 0 \\ 0 & -6 \end{bmatrix}$, $[T^k]_B = ([T]_B)^k = \begin{bmatrix} 4^k & 0 \\ 0 & (-6)^k \end{bmatrix}$

31. Since $[T]_B = \begin{bmatrix} 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $[T(p(x)]_B = \begin{bmatrix} -12 \\ -48 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, then $T(p(x)) = p'''(x) = -12 - 48x$.

32. Since B is the standard basis, then $[T(1)]_B = [1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[T(x)]_B = [2x]_B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, and $[T(x^2)]_B = [3x^2]_B = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$, so $[T]_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

33. $[S]_B^{B'} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $[D]_B^{B'} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, $[D]_B^{B'}[S]_B^{B'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = [T]_B$

34. The linear operator that reflects a vector through the line perpendicular to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, that is reflects across the line $y = -x$, is given by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$, so

$$[T]_B = \left[\begin{bmatrix} -1 \\ -1 \end{bmatrix}_B \begin{bmatrix} -1 \\ 0 \end{bmatrix}_B \right] = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}.$$

35. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the matrix representation for T is $[T]_S = \begin{bmatrix} 0 & -c & b & 0 \\ -b & a-d & 0 & b \\ c & 0 & d-a & -c \\ 0 & c & -b & 0 \end{bmatrix}$.

36. Since $T(\mathbf{v}) = \mathbf{v}$ is the identity map, then

$$[T]_B^{B'} = [\ [T(\mathbf{v_1})]_{B'}\ [T(\mathbf{v_2})]_{B'}\ [T(\mathbf{v_3})]_{B'}] = [\ [\mathbf{v_1}]_{B'}\ [\mathbf{v_2}]_{B'}\ [\mathbf{v_3}]_{B'}] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $[\mathbf{v}]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then $[\mathbf{v}]_{B'} = \begin{bmatrix} b \\ a \\ c \end{bmatrix}$. The matrix $[T]_B^{B'}$ can be obtained from the identity matrix by interchanging the first and second columns.

37.

$$[T]_B = [\ [T(\mathbf{v_1})]_B\ [T(\mathbf{v_2})]_B \dots [T(\mathbf{v_n})\ ]_B = [\ [\mathbf{v_1}]_B\ [\mathbf{v_1}+\mathbf{v_2}]_B \dots [\mathbf{v_{n-1}}+\mathbf{v_n}\ ]_B\ ] = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

Exercise Set 4.5

If $T: V \longrightarrow V$ is a linear operator the matrix representation of T relative to a basis B, denoted $[T]_B$