Exercise Set 4.3

An isomorphism between vector spaces establishes a one-to-one correspondence between the vector spaces. If $T: V \longrightarrow W$ is a one-to-one and onto linear transformation, then T is called an isomorphism. A mapping is one-to-one if and only if $N(T) = \{\mathbf{0}\}$ and is onto if and only if $R(T) = W$. If $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is a basis for V and $T: V \longrightarrow W$ is a linear transformation, then $R(T) = \textbf{span}\{T(\mathbf{v_1}), \ldots, T(\mathbf{v_n})\}$. If in addition, T is one-to-one, then $\{T(\mathbf{v_1}), \ldots, T(\mathbf{v_n})\}$ is a basis for $R(T)$. The main results of Section 4.3 are:

- If V is a vector space with $\dim(V) = n$, then V is isomorphic to $\mathbb{R}^n$.
- If V and W are vector spaces of dimension n, then V and W are isomorphic.

For example, there is a correspondence between the very different vector spaces $\mathcal{P}_3$ and $M_{2\times 2}$. To define the isomorphism, start with the standard basis $S = \{1, x, x^2, x^3\}$ for $\mathcal{P}_3$. Since every polynomial $a+bx+cx^2+dx^3 = a(1) + b(x) + c(x^2) + d(x^3)$ use the coordinate map

$$a + bx + cx^2 + dx^3 \xrightarrow{L_1} [a + bx + cx^2 + dx^3]_S = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ followed by } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \xrightarrow{L_2} \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

so that the composition $L_2(L_1(a + bx + cx^2 + dx^3)) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ defines an isomorphism between $\mathcal{P}_3$ and $M_{2\times 2}$.

Solutions to Exercises

1. Since $N(T) = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$, then T is one-to-one.

2. Since $N(T) = \left\{\begin{bmatrix} -a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R}\right\}$, then T is not one-to-one.

3. Since $N(T) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$, then T is one-to-one.

4. Since $\begin{bmatrix} 2 & -2 & -2 \\ -2 & -1 & -1 \\ -2 & -4 & -1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 2 & -2 & -2 \\ 0 & -3 & -3 \\ 0 & 0 & 3 \end{bmatrix}$, then $N(T) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$, so T is one-to-one.

5. Let $p(x) = ax^2 + bx + c$, so that $p'(x) = 2ax + b$. Then

$$T(p(x)) = 2ax + b - ax^2 - bx - c = -ax^2 + (2a - b)x + (b - c) = 0$$

if and only if $a = 0, 2a - b = 0, b - c = 0$. That is, $p(x)$ is in $N(T)$ if and only if $p(x) = 0$. Hence, T is one-to-one.

6. Let $p(x) = ax^2 + bx + c$, so $T(p(x)) = ax^3 + bx^2 + cx = 0$ if and only if $a = b = c = 0$. Therefore, $N(T)$ consists of only the zero polynomial and hence, T is one-to-one.

7. A vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is in the range of T if the linear system $\begin{cases} 3x - y &= a \\ x + y &= b \end{cases}$ has a solution. Since the linear system is consistent for every vector $\begin{bmatrix} a \\ b \end{bmatrix}$, T is onto $\mathbb{R}^2$. Notice the result also follows from $\det\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\right) = 4$, so the inverse exists.

8. Since $\left[\begin{array}{cc|c} -2 & 1 & a \\ 1 & -1/2 & b \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{cc|c} -2 & 1 & a \\ 0 & 0 & \frac{1}{2}a + b \end{array}\right]$, then a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is in the range of T if and only if $a = -2b$ and hence, T is not onto.

9. Since $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$ is row equivalent to the identity matrix, then the linear operator T is onto $\mathbb{R}^3$.

10. Since

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & a \\ -1 & 1 & 3 & b \\ 1 & 4 & 2 & c \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 2 & 3 & -1 & a \\ 0 & 5 & 5 & a+2b \\ 0 & 0 & 0 & -a-b+c \end{array}\right],$$

then a vector is in the range of T if and only if $-a-b+c=0$ and hence, T is not onto.

11. Since $T(\mathbf{e_1}) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $T(\mathbf{e_2}) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ are two linear independent vectors in $\mathbb{R}^2$, they form a basis.

12. Since $T(\mathbf{e_2}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the set is not a basis.

13. Since $T(\mathbf{e_1}) = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$ and $T(\mathbf{e_2}) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ are two linear independent vectors in $\mathbb{R}^2$, they form a basis.

14. Since $T(\mathbf{e_2}) = 2T(\mathbf{e_1})$, the set is not a basis.

15. Since $T(\mathbf{e_1}) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, $T(\mathbf{e_2}) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $T(\mathbf{e_3}) = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ are three linear independent vectors in $\mathbb{R}^3$, they form a basis.

16. Since $\begin{vmatrix} 2 & 3 & -1 \\ 2 & 6 & 3 \\ 4 & 9 & 2 \end{vmatrix} = 0$, the set is linearly dependent and hence, is not a basis.

17. Since $\begin{vmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -1 & 3/2 \end{vmatrix} = 4$, the set is linearly independent and hence, is a basis.Is a basis.

18. Since $\begin{vmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 0 & -1 & 5 \end{vmatrix} = 6$, the set is linearly independent and hence, is a basis.

19. Since $T(1) = x^2, T(x) = x^2 + x$ and $T(x^2) = x^2 + x + 1$, are three linearly independent polynomials the set is a basis.

20. Since $T(1) = 0$, the set is not a basis.

21. a. Since $\det(A) = \det\left(\begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix}\right) = -3 \neq 0$, then the matrix A is invertible and hence, T is an isomorphism. **b.** $A^{-1} = -\frac{1}{3}\begin{bmatrix} -3 & 0 \\ 2 & 1 \end{bmatrix}$ **c.** Let $\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix}$. To show that $T^{-1}(\mathbf{w}) = A^{-1}\mathbf{w}$, we will show that $A^{-1}(T(\mathbf{w})) = \mathbf{w}$. That is,

$$A^{-1}T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ -2/3 & -1/3 \end{bmatrix}\begin{bmatrix} x \\ -2x-3y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

22. a. Since $\det(A) = \det\left(\begin{bmatrix} -2 & 3 \\ -1 & -1 \end{bmatrix}\right) = 5 \neq 0$, then the matrix A is invertible and hence, T is an isomorphism. **b.** $A^{-1} = \frac{1}{5}\begin{bmatrix} -1 & -3 \\ 1 & -2 \end{bmatrix}$ **c.** Let $\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix}$. To show that $T^{-1}(\mathbf{w}) = A^{-1}\mathbf{w}$, we will show that $A^{-1}(T(\mathbf{w})) = \mathbf{w}$. That is,

$$A^{-1}T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} -1 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -2x+3y \\ -x-y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5x \\ 5y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

23. **a.** Since $\det(A) = \det\left(\begin{bmatrix} -2 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}\right) = -1 \neq 0$, then T is an isomorphism.

b. $A^{-1} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{bmatrix}$ **c.** $A^{-1}T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{bmatrix}\begin{bmatrix} -2x+z \\ x-y-z \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

24. **a.** Since $\det(A) = \det\left(\begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}\right) = 1 \neq 0$, then T is an isomorphism.

b. $A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & 1 \end{bmatrix}$ **c.** $A^{-1}T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2x-y+z \\ -x+y-z \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

25. Since $\begin{vmatrix} -3 & 1 \\ 1 & -3 \end{vmatrix} = 8$, then the matrix mapping T is an isomorphism.

26. Since $\begin{vmatrix} -3 & 1 \\ -3 & 1 \end{vmatrix} = 0$, then the matrix mapping T is not an isomorphism.

27. Since $\begin{vmatrix} 0 & -1 & -1 \\ 2 & 0 & 2 \\ 1 & 1 & -3 \end{vmatrix} = -10$, then the matrix mapping T is an isomorphism.

28. Since $\begin{vmatrix} 1 & 3 & 0 \\ -1 & -2 & -3 \\ 0 & -1 & 3 \end{vmatrix} = 0$, then the matrix mapping T is not an isomorphism.

29. Since $T(cA+B) = (cA+B)^t = cA^t + B^t = cT(A) + T(B)$, T is linear. Since $T(A) = \mathbf{0}$ if and only if $A = \mathbf{0}$, then the null space T is $\{\mathbf{0}\}$, so T is one-to-one. To show that T is onto let B be a matrix in $M_{n\times n}$. If $A = B^t$, then $T(A) = T(B^t) = (B^t)^t = B$, so T is onto. Hence, T is an isomorphism.

30. The transformation is linear and if $p(x) = ax^3 + bx^2 + cx + d$, then

$$T(p(x)) = ax^3 + (3a+b)x^2 + (6a+2b+c)x + (6a+2b+c+d).$$

Hence, the null space consists of only the zero polynomial so T is one-to-one. Since $N(T) = \{0\}$ and $\dim(N(T)) + \dim(R(T)) = 4$, then $\dim(R(T)) = 4$ so that $R(T) = \mathcal{P}_3$ and hence, T is onto.

31. Since $T(kB+C) = A(kB+C)A^{-1} = kABA^{-1} + ACA^{-1} = kT(B) + T(C)$, then T is linear. Since $T(B) = ABA^{-1} = \mathbf{0}$ if and only if $B = \mathbf{0}$ and hence, T is one-to-one. If C is a matrix in $M_{n\times n}$ and $B = A^{-1}CA$, then $T(B) = T(A^{-1}CA) = A(A^{-1}CA)A^{-1} = C$, so T is onto. Hence T is an isomorphism.

32. Define an isomorphism $T : M_{2\times 2} \to \mathbb{R}^4$, by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

33. Define an isomorphism $T : \mathbb{R}^4 \to \mathcal{P}_3$, by

$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = ax^3 + bx^2 + cx + d.$$

34. Define an isomorphism $T : M_{2\times 2} \to \mathcal{P}_3$, by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ax^3 + bx^2 + cx + d$.

35. Since the vector space is given by $V = \left\{\begin{bmatrix} x \\ y \\ x+2y \end{bmatrix} \middle|\, x, y \in \mathbb{R}\right\}$ define an isomorphism $T : V \to \mathbb{R}^2$ by

$$T\left(\begin{bmatrix} x \\ y \\ x+2y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

36. Define an isomorphism $T : \mathcal{P}_2 \to, V$ by $T(ax^2 + bx + c) = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$.