is a linear transformation. The null space of T, denoted by $N(T)$, is the null space of the matrix, $N(A) = \{\mathbf{x} \in \mathbb{R}^3 \mid A\mathbf{x} = \mathbf{0}\}$. Since

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix},$$

the range of T, denoted by $R(T)$ is the column space of A, $\mathbf{col}(A)$. Since

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 3 \\ 2 & 0 & 3 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -6 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions given by $x_1 = -\frac{3}{2}x_3, x_2 = \frac{1}{2}x_3$, and x_3 a free variable. So the null space is $\left\{ t\begin{bmatrix} -3/2 \\ 1/2 \\ 1 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$, which is a line that passes through the origin in three space. Also since the pivots in the reduced matrix are in columns one and two, a basis for the range is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right\}$ and hence, the range is a plane in three space. Notice that in this example, $3 = \dim(\mathbb{R}^3) = \dim(R(T)) + \dim(N(T))$. This is a fundamental theorem that if $T : V \longrightarrow W$ is a linear transformation defined on finite dimensional vector spaces, then

$$\dim(V) = \dim(R(T)) + \dim(N(T)).$$

If the mapping is given as a matrix product $T(\mathbf{v}) = A\mathbf{v}$ such that A is a $m \times n$ matrix, then this result is written as

$$n = \mathbf{rank}(A) + \mathbf{nullity}(A).$$

A number of useful statements are added to the list of equivalences concerning $n \times n$ linear systems:

A is invertible $\Leftrightarrow A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \Leftrightarrow A\mathbf{x} = \mathbf{0}$ has only the trivial solution
$\Leftrightarrow A$ is row equivalent to $I \Leftrightarrow \det(A) \neq 0 \Leftrightarrow$ the column vectors of A are linearly independent
$\Leftrightarrow$ the column vectors of A span $\mathbb{R}^n \Leftrightarrow$ the column vectors of A are a basis for $\mathbb{R}^n$
$\Leftrightarrow \mathbf{rank}(A) = n \Leftrightarrow R(A) = \mathbf{col}(A) = \mathbb{R}^n \Leftrightarrow N(A) = \{\mathbf{0}\} \Leftrightarrow \mathbf{row}(A) = \mathbb{R}^n$
$\Leftrightarrow$ the number of pivot columns in the row echelon form of A is n.

Solutions to Exercises

1. Since $T(\mathbf{v}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{v}$ is in $N(T)$.

2. Since $T(\mathbf{v}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{v}$ is in $N(T)$.

3. Since $T(\mathbf{v}) = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$, $\mathbf{v}$ is not in $N(T)$.

4. Since $T(\mathbf{v}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{v}$ is in $N(T)$.

5. Since $p'(x) = 2x - 3$ and $p''(x) = 2$, then $T(p(x)) = 2x$, so $p(x)$ is not in $N(T)$.

6. Since $p'(x) = 5$ and $p''(x) = 0$, then $T(p(x)) = 0$, so $p(x)$ is in $N(T)$.

7. Since $T(p(x)) = -2x$, then $p(x)$ is not in $N(T)$.

8. Since $T(p(x)) = 0$, then $p(x)$ is in $N(T)$.

9. Since $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 0 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ there are infinitely many vectors that are mapped to $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$. For example, $T\left(\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and hence, $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ is in $R(T)$.

10. Since $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 4 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & \\ 0 & 0 & 0 & 1 \end{array}\right]$ the linear system is inconsistent, so the vector $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ is not in $R(T)$.

11. Since $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 2 & 1 & 3 & 1 \\ 1 & -1 & 3 & -2 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$, the linear system is inconsistent, so the vector $\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$ is not in $R(T)$.

12. Since $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 2 & 1 & 3 & -5 \\ 1 & -1 & 3 & -1 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ there are infinitely many vectors that are mapped to $\begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix}$ and hence, the vector $\begin{bmatrix} -2 \\ -5 \\ -1 \end{bmatrix}$ is in $R(T)$.

13. The matrix A is in $R(T)$.

14. The matrix A is not in $R(T)$.

15. The matrix A is not in $R(T)$.

16. The matrix A is in $R(T)$.

17. A vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is in the null space, if and only if $3x + y = 0$ and $y = 0$. That is, $N(T) = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$. Hence, the null space has dimension 0, so does not have a basis.

18. A vector is in the null space if and only if $\begin{cases} -x + y &= 0 \\ x - y &= 0 \end{cases}$, that is $x = y$. Therefore, $N(T) = \left\{\begin{bmatrix} a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R}\right\}$ and hence, a basis is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

19. Since $\begin{bmatrix} x + 2z \\ 2x + y + 3z \\ x - y + 3z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ if and only if $x = -2z$ and $y = z$ every vector in the null space has the form $\begin{bmatrix} -2z \\ z \\ z \end{bmatrix}$. Hence, a basis for the null space is $\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$.

20. Since $\begin{bmatrix} -2 & 2 & 2 \\ 3 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$, then $N(T) = \left\{ t\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \middle| \, t \in \mathbb{R}\right\}$ and a basis for the null space is $\left\{\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}\right\}$.

21. Since $N(T) = \left\{\begin{bmatrix} 2s + t \\ s \\ t \end{bmatrix} \middle| \, s, t \in \mathbb{R}\right\}$, a basis for the null space is $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

22. A basis for the null space is $\left\{\begin{bmatrix} -5 \\ 6 \\ 1 \\ 0 \end{bmatrix}\right\}$.

23. Since $T(p(x)) = 0$ if and only if $p(0) = 0$ a polynomial is in the null space if and only if it has the form $ax^2 + bx$. A basis for the null space is $\{x, x^2\}$.

24. If $p(x) = ax^2 + bx + c$, then $p'(x) = 2ax + b$ and $p''(x) = 2a$, so $T(p(x)) = 0$ if and only if $a = 0$. A basis for the null space is $\{1, x\}$.

25. Since $\det\left(\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}\right) = -5$, the column vectors of the matrix are a basis for the column space of the matrix. Since the column space of the matrix is $R(T)$, then a basis for the range of T is $\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}$.

26. Since $\begin{bmatrix} 1 & -2 & -3 & 1 & 5 \\ 3 & -1 & 1 & 0 & 4 \\ 1 & 1 & 3 & 1 & 2 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ and the pivots are in columns one, two, and four, then a basis for the column space of A and hence, for $R(T)$, is $\left\{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

27. Since the range of T is the xy-plane in $\mathbb{R}^3$, a basis for the range is $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

28. Since $\begin{bmatrix} x - y + 3z \\ x + y + z \\ -x + 3y - 5z \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + z\begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}$ and the three vectors are linearly independent, then a basis for $R(T)$ is $\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}\right\}$.

29. Since $R(T) = \mathcal{P}_2$, then a basis for the range is $\{1, x, x^2\}$.

30. Since

$$R(T) = \{p(x) \mid p(x) = ax^2+bx+a = a(x^2+1)+bx\},$$

then a basis for $R(T)$ is $\{x, x^2 + 1\}$.

31. a. The vector $\mathbf{w}$ is in the range of T if the linear system

$$c_1\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}$$

has a solution. But $\left[\begin{array}{ccc|c} -2 & 0 & -2 & -6 \\ 1 & 1 & 2 & 5 \\ 1 & -1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} -2 & 0 & -2 & -6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right]$, so that the linear system is inconsistent. Hence, $\begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}$ is not in $R(t)$.

b. Since $\begin{vmatrix} -2 & 0 & -2 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{vmatrix} = 0$, the column vectors are linearly dependent. To trim the vectors to a basis for the range, we have that $\begin{bmatrix} -2 & 0 & -2 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Since the pivots are in columns one and two, a basis for the range is $\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$. **c.** Since $\dim(N(T)) + \dim(R(T)) = \dim(\mathbb{R}^3) = 3$ and $\dim(R(T)) = 2$, then $\dim(N(T)) = 1$.

32. a. The vector $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is in $R(T)$. **b.** $\left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}\right\}$ **c.** Since $\dim(N(T)) + \dim(R(T)) = 3$ and $\dim(R(T)) = 3$, then $\dim(N(T)) = 0$.

33. a. The polynomial $2x^2 - 4x + 6$ is not in $R(T)$. **b.** Since the null space of T is the set of all constant functions, then $\dim(N(T)) = 1$ and hence, $\dim(R(T)) = 2$. A basis for the range is $\{T(x), T(x^2)\} = \{-2x + 1, x^2 + x\}$.

34. a. The polynomial $x^2 - x - 2$ is not in $R(T)$. **b.** Since the null space of T is the set of all polynomials of the form ax^2, then $\dim(N(T)) = 1$ and hence, $\dim(R(T)) = 2$. A basis for the range is $\{T(1), T(x)\} = \{x^2, x - 1\}$.

35. Any linear transformations that maps three space to the entire xy-plane will work. For example, the mapping to the xy-plane is $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$.

36. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$, by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ 0 \end{bmatrix}$. Then $N(T) = \left\{\begin{bmatrix} x \\ 0 \end{bmatrix} \middle| \, x \in \mathbb{R}\right\} = R(T)$.

37. a. The range $R(T)$ is the subspace of $\mathcal{P}_n$ consisting of all polynomials of degree $n - 1$ or less. **b.** $\dim(R(T)) = n$ **c.** Since $\dim(R(T)) + \dim(N(T)) = \dim(\mathcal{P}_n) = n + 1$, then $\dim(N(T)) = 1$.

38. A polynomial is in the null space provided it has degree $k - 1$ or less. Hence $\dim(N(T)) = k$.

39. a. $\dim(R(T)) = 2$ **b.** $\dim(N(T)) = 1$

40. Since $\dim(V) = \dim(N(T)) + \dim(R(T)) = 2\dim(N(T))$, then the dimension of V is an even number.

41. If $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $T(B) = AB - BA = \begin{bmatrix} 0 & 2b \\ -2c & 0 \end{bmatrix}$, so that $N(T) = \left\{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \middle| \, a, d \in \mathbb{R}\right\} = \left\{a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \middle| \, a, d \in \mathbb{R}\right\}$. Hence a basis for $N(T)$ is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

42. If B is an $n \times n$ matrix, then $T(B^t) = (B^t)^t = B$ and hence, $R(T) = M_{n\times n}$.

43. a. Notice that $(A + A^t)^t = A^t + A = A + A^t$, so that the range of T is a subset of the symmetric matrices. Also if B is any symmetric matrix, then $T\left(\frac{1}{2}B\right) = \frac{1}{2}B + \frac{1}{2}B^t = B$. Therefore, $R(T)$ is the set of all symmetric matrices. **b.** Since a matrix A is in $N(T)$ if and only if $T(A) = A + A^t = \mathbf{0}$, which is if and only if $A = -A^t$, then the null space of T is the set of skew-symmetric matrices.

44. a. Notice that $(A - A^t)^t = A^t - A = -(A - A^t)$, so that the range of T is a subset of the skew-symmetric matrices. Also if B is any skew-symmetric matrix, then $T\left(\frac{1}{2}B\right) = \frac{1}{2}B - \frac{1}{2}B^t = B$. Therefore, $R(T)$ is the set of all skew-symmetric matrices. **b.** Since a matrix A is in $N(T)$ if and only if $T(A) = A - A^t = \mathbf{0}$, which is if and only if $A = A^t$, then the null space of T is the set of symmetric matrices.

45. If the matrix A is invertible and B is any $n \times n$ matrix, then $T(A^{-1}B) = A(A^{-1}B) = B$, so $R(T) = M_{n\times n}$.

46. a. A basis for the range of T consists of the column vectors of A corresponding to the pivot columns of the echelon form of A. Any zero rows of A correspond to diagonal entries that are 0, so the echelon form of A will have pivot columns corresponding to each nonzero diagonal term. Hence, the range of T is spanned by the nonzero column vectors of A and the number of nonzero vectors equals the number of pivot columns of A. **b.** Since $\dim(N(T)) = n = \dim(R(T))$, then the dimension of the null space of T is the number of zeros on the diagonal.