4 Linear Transformations

Exercise Set 4.1

A linear transformation is a special kind of function (or mapping) defined from one vector space to another. To verify $T : V \longrightarrow W$ is a linear transformation from V to W, then we must show that T satisfies the two properties

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(c\mathbf{u}) = cT(\mathbf{u})$$

or equivalently just the one property

$$T(\mathbf{u}+c\mathbf{v}) = T(\mathbf{u}) + cT(\mathbf{v}).$$

The addition and scalar multiplication in $T(\mathbf{u}+c\mathbf{v})$ are the operations defined on V and in $T(\mathbf{u}) + cT(\mathbf{v})$ the operations defined on W. For example, $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ x-y \end{bmatrix} \quad \text{is a linear transformation.}$$

To see this compute the combination $\mathbf{u}+c\mathbf{v}$ of two vectors and then apply T. Notice that the definition of T requires the input of only one vector, so to apply T first simplify the expression. Then we need to consider

$$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+cx_2 \\ y_1+cy_2 \end{bmatrix}\right).$$

Next apply the definition of the mapping resulting in a vector with two components. To find the first component add the first component of the input vector to twice the second component and for the second component of the result subtract the components of the input vector. So

$$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+cx_2 \\ y_1+cy_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1+cx_2)+2(y_1+cy_2) \\ (x_1+cx_2)-(y_1+cy_2) \end{bmatrix}.$$

The next step is to rewrite the output vector in the correct form. This gives

$$\begin{aligned} \begin{bmatrix} (x_1+cx_2)+2(y_1+cy_2) \\ (x_1+cx_2)-(y_1+cy_2) \end{bmatrix} &= \begin{bmatrix} (x_1+2y_1)+c(x_2+2y_2) \\ (x_1-y_1)+c(x_2-y_2) \end{bmatrix} \\ &= \begin{bmatrix} x_1+2y_1 \\ x_1-y_1 \end{bmatrix} + c\begin{bmatrix} x_2+2y_2 \\ x_2-y_2 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + cT\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right), \end{aligned}$$

and hence T is a linear transformation. On the other hand a mapping defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y \end{bmatrix} \quad \text{is not a linear transformation}$$

since, for example,

$$\begin{aligned} T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) &= \begin{bmatrix} x_1+x_2+1 \\ y_1+y_2 \end{bmatrix} \\ &\neq T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2+1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2+2 \\ y_1+y_2 \end{bmatrix} \end{aligned}$$

for all pairs of vectors. Other useful observations made in Section 4.1 are:

- For every linear transformation $T(\mathbf{0}) = \mathbf{0}$.

- If A is an $m \times n$ matrix, then $T(\mathbf{v}) = A\mathbf{v}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- $T(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}) = c_1T(\mathbf{v_1}) + c_2T(\mathbf{v_2}) + \cdots + c_nT(\mathbf{v_n})$

The third property can be used to find the image of a vector when the action of a linear transformation is known only on a specific set of vectors, for example on the vectors of a basis. For example, suppose that $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear transformation and

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}.$$

Then the image of an arbitrary input vector can be found since $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$.

For example, let's find the image of the vector $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$. The first step is to write the input vector in terms of the basis vectors, so

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Then use the linearity properties of T to obtain

$$T\left(\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}\right) = T\left(-\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = -T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$$
$$= -\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}.$$

Solutions to Exercises

1. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$ and c a scalar. Since

$$T(\mathbf{u} + c\mathbf{v}) = T\left(\begin{bmatrix} u_1 + cv_1 \\ u_2 + cv_2 \end{bmatrix}\right) = \begin{bmatrix} u_2 + cv_2 \\ u_1 + cv_1 \end{bmatrix} = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} + c\begin{bmatrix} v_2 \\ v_1 \end{bmatrix} = T(\mathbf{u}) + cT(\mathbf{v}),$$

then T is a linear transformation.

2. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$ and c a scalar. Then

$$T(\mathbf{u} + c\mathbf{v}) = T\left(\begin{bmatrix} u_1 + cv_1 \\ u_2 + cv_2 \end{bmatrix}\right) = \begin{bmatrix} (u_1 + cv_1) + (u_2 + cv_2) \\ (u_1 + cv_1) - (u_2 + cv_2) + 2 \end{bmatrix}$$

and

$$T(\mathbf{u}) + cT(\mathbf{v}) = \begin{bmatrix} (u_1 + cv_1) + (u_2 + cv_2) \\ (u_1 + cv_1) - (u_2 + cv_2) + 4 \end{bmatrix}.$$

For example, if $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $c = 1$, then $T(\mathbf{u} + \mathbf{v}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $T(\mathbf{u}) + T(\mathbf{v}) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Hence, T is not a linear transformation.

3. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$. Since

$$T(\mathbf{u}+\mathbf{v}) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} u_1+v_1 \\ u_1^2+2u_2v_2+v_2^2 \end{bmatrix} \text{ and } \begin{bmatrix} u_1+v_1 \\ u_2^2+v_2^2 \end{bmatrix} = T(\mathbf{u})+T(\mathbf{v}),$$

which do not agree for all vectors, T is not a linear transformation.

4. Since

$$T(\mathbf{u}+c\mathbf{v}) = T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} u_1+cv_1 \\ u_2+cv_2 \end{bmatrix}\right) = \left(\begin{bmatrix} 2(u_1+cv_1)-(u_2+cv_2) \\ u_1+cv_1+3(u_2+cv_2) \end{bmatrix}\right)$$

and

$$T(\mathbf{u})+cT(\mathbf{v}) = \begin{bmatrix} 2u_1-u_2 \\ u_1+3u_2 \end{bmatrix} + c\begin{bmatrix} 2v_1-v_2 \\ v_1+3v_2 \end{bmatrix} = \left(\begin{bmatrix} 2(u_1+cv_1)-(u_2+cv_2) \\ u_1+cv_1+3(u_2+cv_2) \end{bmatrix}\right),$$

then T is a linear transformation.

5. Since $T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} u_1+cv_1 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + cT\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$, for all pairs of vectors and scalars c, T is a linear transformation.

6. Since

$$T(\mathbf{u}+c\mathbf{v}) = T\left(\begin{bmatrix} u_1+cv_1 \\ u_2+cv_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{(u_1+cv_1)+(u_2+cv_2)}{2} \\ \frac{(u_1+cv_1)+(u_2+cv_2)}{2} \end{bmatrix} = T(\mathbf{u})+cT(\mathbf{v}),$$

then T is a linear transformation.

7. Since $T(x+y) = T(x)+T(y)$, if and only if at least one of x or y is zero, T is not a linear transformation.

8. Since T describes a straight line passing through the origin, then T defines a linear transformation.

9. Since $T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = c^2(x^2+y^2) = cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = c(x^2+y^2)$ if and only if $c=1$ or $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, T is not a linear transformation.

10. Since T is the identity mapping on the first two coordinates, then T is a linear transformation.

11. Since $T(\mathbf{0}) \neq \mathbf{0}$, T is not a linear transformation.

12. Since $\cos 0 = 1$, then $T(\mathbf{0}) \neq \mathbf{0}$ and hence, T is not a linear transformation.

13. Since

$$\begin{aligned} T(p(x)+q(x)) &= 2(p''(x)+q''(x)) - 3(p'(x)+q'(x)) + (p(x)+q(x)) \\ &= (2p''(x)-3p'(x)+p(x)) + (2q''(x)-3q'(x)+q(x)) = T(p(x))+T(q(x)) \end{aligned}$$

and similarly, $T(cp(x)) = cT(p(x))$ for all scalars c, T is a linear transformation.

14. Since $T(p(x)+q(x)) = p(x)+q(x)+x$ and $T(p(x))+T(q(x)) = p(x)+q(x)+2x$ these will not always agree and hence, T is not a linear transformation. Also $T(0) = x$ and hence the zero polynomial is not mapped to the zero polynomial.

15. If A is a 2×2 matrix, then $\det(cA) = c^2\det(A)$, then for example, if $c=2$ and $\det(A)\neq 0$, we have that $T(2A) = 4T(A) \neq 2T(A)$ and hence, T is not a linear transformation.

16. Since $T(A+cB) = A+cB+(A+cB)^t = A+cB+A^t+cB^t = T(A)+cT(B)$, then T is a linear transformation.

17. **a.** $T(\mathbf{u}) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$; $T(\mathbf{v}) = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ **b.** $T(\mathbf{u}+\mathbf{v}) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = T(\mathbf{u})+T(\mathbf{v})$

c. The mapping T is a linear transformation.

18. a. $T(\mathbf{u}) = x^2 - 7x + 9$; $T(\mathbf{v}) = -x + 1$ **b.** $T(\mathbf{u}+\mathbf{v}) = T(x^2 - 4x) = x^2 - 8x + 10 = T(\mathbf{u}) + T(\mathbf{v})$
c. The mapping T is a linear transformation.

19. a. $T(\mathbf{u}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; $T(\mathbf{v}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ **b.** $T(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq T(\mathbf{u}) + T(\mathbf{v}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$
c. By part (b), T is not a linear transformation.

20. a. $T(\mathbf{u}) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$; $T(\mathbf{v}) = \begin{bmatrix} -3/4 \\ 0 \end{bmatrix}$ **b.** $T(\mathbf{u}+\mathbf{v}) = T\left(\begin{bmatrix} 1/2 \\ 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -3/4 \\ 5 \end{bmatrix} = T(\mathbf{u}) + T(\mathbf{v})$
c. The mapping T is not a linear transformation. For example, if the first coordinates of two vectors are 1 and 2, respectively, then $T\left(\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 8 \\ x+y+z+w \end{bmatrix}$ but
$T\left(\begin{bmatrix} 1 \\ x \\ y \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x+y \end{bmatrix} + \begin{bmatrix} 3 \\ z+w \end{bmatrix} = \begin{bmatrix} 3 \\ x+y+z+w \end{bmatrix}$. Alternatively, $T(\mathbf{0}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

21. Since $\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and T is a linear transformation, we have that

$$T\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) - 3T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ -9 \end{bmatrix}.$$

22. Since $\begin{bmatrix} 1 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and T is a linear operator, then

$$T\left(\begin{bmatrix} 1 \\ 7 \\ 5 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + 7T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + 5T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 14 \\ 0 \\ 7 \end{bmatrix} + \begin{bmatrix} 5 \\ -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ -6 \\ 12 \end{bmatrix}.$$

23. Since $T(-3 + x - x^2) = T(-3(1) + 1(x) + (-1)x^2$ and T is a linear operator, then
$T(-3 + x - x^2) = -3(1+x) + (2+x^2) - (x - 3x^2) = -1 - 4x + 4x^2$.

24.

$$T\left(\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\right) = 2T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) + 3T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$$
$$= 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix}$$

25. Since $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$ and T is a linear operator, then it is possible to find $T(\mathbf{v})$ for every vector in $\mathbb{R}^2$. In particular, $T\left(\begin{bmatrix} 3 \\ 7 \end{bmatrix}\right) = T\left(7\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 22 \\ -11 \end{bmatrix}$.

26. a. $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, T(\mathbf{e}_3) = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$

b. $T(3\mathbf{e}_1 - 4\mathbf{e}_2 + 6\mathbf{e}_3) = 3T(\mathbf{e}_1) - 4T(\mathbf{e}_2) + 6T(\mathbf{e}_3) = 3\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + 6\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 20 \\ 3 \end{bmatrix}$

27. **a.** Since the polynomial $2x^2-3x+2$ cannot be written as a linear combination of x^2, $-3x$, and $-x^2+3x$, from the given information the value of $T(2x^2-3x+2)$ can not be determined. That is, the equation $c_1x^2+c_2(-3x)+c_3(-x^2+3x)=2x^2-2x+1$ is equivalent to $(c_1-c_3)x^2+(-3c_2+3c_3)x=2x^2-3x+2$, which is not possible. **b.** $T(3x^2-4x)=T\left(3x^2+\frac{4}{3}(-3x)\right)=3T(x^2)+\frac{4}{3}T(-3x)=\frac{4}{3}x^2+6x-\frac{13}{3}$.

28. **a.** Since $\begin{bmatrix}2\\-5\\0\end{bmatrix}=7\begin{bmatrix}1\\0\\0\end{bmatrix}-5\begin{bmatrix}1\\1\\0\end{bmatrix}$, then $T\left(\begin{bmatrix}2\\-5\\0\end{bmatrix}\right)=7\begin{bmatrix}-1\\2\\3\end{bmatrix}+5\begin{bmatrix}2\\-2\\1\end{bmatrix}=\begin{bmatrix}3\\4\\26\end{bmatrix}$.

b. Since the $\begin{vmatrix}1&1&1\\0&1&3\\0&0&0\end{vmatrix}$ has a row of zeros its value is 0, so the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}1\\1\\0\end{bmatrix}$, and $\begin{bmatrix}1\\3\\0\end{bmatrix}$ are linearly dependent. Therefore, it is not possible to determine $T(\mathbf{v})$ for every vector in $\mathbb{R}^3$.

29. **a.** If $A=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$, then $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=A\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-x\\-y\end{bmatrix}$. **b.** $T(\mathbf{e}_1)=\begin{bmatrix}-1\\0\end{bmatrix}$ and $T(\mathbf{e}_2)=\begin{bmatrix}0\\-1\end{bmatrix}$. Observe that these are the column vectors of A.

30. **a.** Let $A=\begin{bmatrix}1&-2\\3&1\\0&2\end{bmatrix}$. **b.** $T(\mathbf{e}_1)=\begin{bmatrix}1\\3\\0\end{bmatrix}$, $T(\mathbf{e}_2)=\begin{bmatrix}-2\\1\\2\end{bmatrix}$

31. A vector $\begin{bmatrix}x\\y\\z\end{bmatrix}$ in $\mathbb{R}^3$ is mapped to the zero vector if and only if $\begin{bmatrix}x+y\\x-y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, that is, if and only if $x=y=0$. Consequently, $T\left(\begin{bmatrix}0\\0\\z\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}$, for all $z\in\mathbb{R}$.

32. Since $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}0\\0\\0\end{bmatrix}$ if and only if $\begin{cases}x+2y+z&=0\\-x+5y+z&=0\end{cases}$ and $\begin{bmatrix}1&2&1\\-1&5&1\end{bmatrix}\xrightarrow{\text{reduces to}}\begin{bmatrix}1&2&1\\0&7&2\end{bmatrix}$, then all vectors that are mapped to the zero vector have the form $\begin{bmatrix}-\frac{3}{7}z\\-\frac{2}{7}z\\z\end{bmatrix}$, $z\in\mathbb{R}$.

33. **a.** Since

$$\left[\begin{array}{ccc|c}1&-1&2&0\\2&3&-1&0\\-1&2&-2&0\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&-1&2&0\\0&5&-5&0\\0&0&1&0\end{array}\right],$$

the zero vector is the only vector in $\mathbb{R}^3$ such that $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}$.

b. Since

$$\left[\begin{array}{ccc|c}1&-1&2&7\\2&3&-1&-6\\-1&2&-2&-9\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&-2\\0&0&1&2\end{array}\right],\text{ then }T\left(\begin{bmatrix}1\\-2\\2\end{bmatrix}\right)=\begin{bmatrix}7\\-6\\-9\end{bmatrix}.$$

34. **a.** Since $T(ax^2+bx+c)=(2ax+b)-c$, then $T(p(x))=0$ if and only if $2a=0$ and $b-a=0$. Hence, $T(p(x))=0$ if and only if $p(x)=bx+b=b(x+1)$ for any real number b. **b.** Let $p(x)=3x^2-3x$, then $T(p(x))=6x-3$. As a second choice let $q(x)=3x^2-5x-2$, so $q(0)=-2$ and $T(q(x))=q'(x)-q(0)=6x-5+2=6x-3$. **c.** The mapping T is a linear operator.

35. Since $T(c\mathbf{v}+\mathbf{w})=\begin{bmatrix}cT_1(\mathbf{v})+T_1(\mathbf{w})\\cT_2(\mathbf{v})+T_2(\mathbf{w})\end{bmatrix}=c\begin{bmatrix}T_1(\mathbf{v})\\T_2(\mathbf{v})\end{bmatrix}+\begin{bmatrix}T_1(\mathbf{w})\\T_2(\mathbf{w})\end{bmatrix}=cT(\mathbf{v})+T(\mathbf{w})$, then T is a linear transformation.

36. Since $\mathbf{tr}(A+B) = \mathbf{tr}(A) + \mathbf{tr}(B)$ and $\mathbf{tr}(cA) = c\mathbf{tr}(A)$, then $T(A+cB) = T(A) + cT(B)$ and hence, T is a linear transformation.

37. Since $T(kA+C) = (kA+C)B - B(kA+C) = kAB - kBA + CB - BC = kT(A) + T(C)$, then T is a linear operator.

38. Since $T(x+y) = m(x+y) + b$ and $T(x) + T(y) = m(x+y) + 2b$, then $T(x+y) = T(x) + T(y)$ if and only if $b = 0$. If $b = 0$, then we also have that $T(cx) = cmx = cT(x)$. Hence, T is a linear operator if and only if $b = 0$.

39. a. Using the properties of the Riemann Integral, we have that

$$T(cf+g) = \int_0^1 (cf(x) + g(x))\,dx = \int_0^1 cf(x)dx + \int_0^1 g(x)dx = c\int_0^1 f(x)dx + \int_0^1 g(x)dx = cT(f) + T(g)$$

so T is a linear operator. **b.** $T(2x^2 - x + 3) = \frac{19}{6}$

40. Since T is a linear operator, $T(\mathbf{u}) = \mathbf{w}$, and $T(\mathbf{v}) = \mathbf{0}$, then $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{w})$.

41. Since $\{\mathbf{v}, \mathbf{w}\}$ is linear independent $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{w} \neq \mathbf{0}$. Hence, if either $T(\mathbf{v}) = 0$ or $T(\mathbf{w}) = 0$, then the conclusion holds. Now assume that $T(\mathbf{v})$ and $T(\mathbf{w})$ are linearly dependent and not zero. So there exist scalars a and b, not both 0, such that $aT(\mathbf{v}) + bT(\mathbf{w}) = \mathbf{0}$. Since $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, then $a\mathbf{v} + b\mathbf{w} \neq \mathbf{0}$. Hence, since T is linear, then $aT(\mathbf{v}) + bT(\mathbf{w}) = T(a\mathbf{v} + b\mathbf{w}) = \mathbf{0}$, and we have shown that $T(\mathbf{u}) = \mathbf{0}$ has a nontrivial solution.

42. Since $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is linearly dependent there are scalars $c_1, c_2, \ldots, c_n$, not all zero, such that $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{0}$. Since T is a linear operator, then

$$T(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}) = c_1T(\mathbf{v_1}) + c_2T(\mathbf{v_2}) + \cdots + c_nT(\mathbf{v_n}) = T(\mathbf{0}) = \mathbf{0}.$$

Therefore, $\{T(\mathbf{v_1}), \ldots, T(\mathbf{v_n})\}$ is linearly dependent.

43. Let $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$ in $\mathbb{R}^3$.

44. Let $\mathbf{v}$ be a vector in V. Since $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is a basis there are scalars $c_1, c_2, \ldots, c_n$ such that $\mathbf{v} = c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n}$. Since T_1 and T_2 are linear operators, then

$$T_1(c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n}) = c_1T_1(\mathbf{v_1}) + \cdots + c_nT_1(\mathbf{v_n}) \quad \text{and} \quad T_2(c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n}) = c_1T_2(\mathbf{v_1}) + \cdots + c_nT_2(\mathbf{v_n}).$$

Since $T_1(\mathbf{v_i}) = T_2(\mathbf{v_i})$, for each $i = 1, 2, \ldots, n$, then $T_1(\mathbf{v}) = T_2(\mathbf{v})$.

45. We use the definitions $(S+T)(\mathbf{v}) = S(\mathbf{u}) + T(\mathbf{u})$ and $(cT)(\mathbf{u}) = c(T(\mathbf{u}))$ for addition and scalar multiplication of linear transformations. Then $\mathcal{L}(U,V)$ with these operations satisfy all ten of the vector space axioms. For example, $\mathcal{L}(U,V)$ is closed under addition since

$$\begin{aligned}(S+T)(c\mathbf{u}+\mathbf{v}) &= S(c\mathbf{u}+\mathbf{v}) + T(c\mathbf{u}+\mathbf{v})\\ &= cS(\mathbf{u}) + S(\mathbf{v}) + cT(\mathbf{u}) + T(\mathbf{v})\\ &= cS(\mathbf{u}) + cT(\mathbf{u}) + S(\mathbf{v}) + T(\mathbf{v})\\ &= c(S+T)(\mathbf{u}) + (S+T)(\mathbf{v}).\end{aligned}$$

Exercise Set 4.2

If $T : V \longrightarrow W$ is a linear transformation, then the null space is the subspace of all vectors in V that are mapped to the zero vector in W and the range of T is the subspace of W consisting of all images of vectors from V. Any transformation defined by a matrix product is a linear transformation. For example, $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$