$S \cap T = \left\{ s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \middle| \; s \in \mathbb{R} \right\}$, then $\dim(S \cap T) = 1$.

Exercise Set 3.4

If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an ordered basis for a vector space, then for each vector $\mathbf{v}$ there are scalars $c_1, c_2, \ldots, c_n$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$. The unique scalars are called the coordinates of the vector relative to the ordered basis B, written as

$$[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

If B is one of the standard bases of $\mathbb{R}^n$, then the coordinates of a vector are just the components of the vector. For example, since every vector in $\mathbb{R}^3$ can be written as

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

the coordinates relative to the standard basis are $[\mathbf{v}]_B = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. To find the coordinates relative to an ordered basis solve the usual vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{v}$$

for the scalars $c_1, c_2, \ldots, c_n$. The order in which the vectors are given makes in difference when defining coordinates. For example, if $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $B' = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, then

$$[\mathbf{v}]_B = \left[\begin{bmatrix} x \\ y \end{bmatrix} \right]_B = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad [\mathbf{v}]_{B'} = \left[\begin{bmatrix} x \\ y \end{bmatrix} \right]_{B'} = \begin{bmatrix} y \\ x \end{bmatrix}.$$

Given the coordinates relative to one basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to find the coordinates of the same vector relative to a second basis $B' = \{\mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$ a transition matrix can be used. To determine a transition matrix:

- Find the coordinates of each vector in B relative to the basis B'.
- Form the transition matrix where the column vectors are the coordinates found in the first step. That is,
$$[I]_B^{B'} = [\, [\mathbf{v}_1]_{B'} \; [\mathbf{v}_2]_{B'} \ldots [\mathbf{v}_n]_{B'} \,].$$
- The coordinates of $\mathbf{v}$ relative to B' given the coordinates relative B are given by the formula
$$[\mathbf{v}]_{B'} = [I]_B^{B'}[\mathbf{v}]_B.$$

The transition matrix that changes coordinates from B' to B is given by

$$[I]_{B'}^{B} = ([I]_B^{B'})^{-1}.$$

Let $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and $B' = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right\}$ be two bases for $\mathbb{R}^2$. The steps for finding the transition matrix from B to B' are:

- Since
$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & -1 \\ 2 & -1 & 1 & 1 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{cc|cc} 1 & -2 & 1 & -1 \\ 0 & 3 & -1 & 3 \end{array}\right],$$
so the coordinates of the two vectors in B relative to B' are $\left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]_{B'} = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}$ and $\left[\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]_{B'} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- $[I]_B^{B'} = \begin{bmatrix} 1/3 & 1 \\ -1/3 & 1 \end{bmatrix}$
- As an example, $\left[\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right]_{B'} = \begin{bmatrix} 1/3 & 1 \\ -1/3 & 1 \end{bmatrix}\left[\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right]_B = \begin{bmatrix} 1/3 & 1 \\ -1/3 & 1 \end{bmatrix}\begin{bmatrix} 1/2 \\ -5/2 \end{bmatrix} = \begin{bmatrix} -7/3 \\ -8/3 \end{bmatrix}$.

Solutions to Exercises

1. The coordinates of $\begin{bmatrix} 8 \\ 0 \end{bmatrix}$, relative to the basis B are the scalars c_1 and c_2 such that $c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$. The vector equation yields the linear system $\begin{cases} 3c_1 - 2c_2 & = 8 \\ c_1 + 2c_2 & = 0 \end{cases}$, which has the unique solution $c_1 = 2$, and $c_2 = -1$. Hence, $[\mathbf{v}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

2. Since $\begin{bmatrix} -2 & -1 & -2 \\ 4 & 1 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3 \end{bmatrix}$, then $[\mathbf{v}]_B = \begin{bmatrix} -1/2 \\ 3 \end{bmatrix}$.

3. To find the coordinates we form and row reduce the matrix
$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ -1 & -1 & 0 & -1 \\ 2 & 1 & 2 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right], \text{ so that } [\mathbf{v}]_B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

4. Since $\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1/2 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$, then $[\mathbf{v}]_B = \begin{bmatrix} 1/2 \\ -1 \\ 2 \end{bmatrix}$.

5. Since $c_1 + c_2(x-1) + c_3x^2 = 3 + 2x - 2x^2$ if and only if $c_1 - c_2 = 3, c_2 = 2$, and $c_3 = -2$, we have that $[\mathbf{v}]_B = \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}$.

6. The equation $c_1(x^2 + 2x + 2) + c_2(2x + 3) + c_3(-x^2 + x + 1) = 8 + 6x - 3x^2$ gives the linear system, in matrix form,
$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 2 & 2 & 1 & 6 \\ 1 & 0 & -1 & -3 \end{array}\right], \xrightarrow{\text{which reduces to}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 8/3 \end{array}\right]$$
so $[\mathbf{v}]_B = \begin{bmatrix} -1/3 \\ 2 \\ 8/3 \end{bmatrix}$.

7. Since $c_1\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}$ if and only if

$\begin{cases} c_1 + c_3 + c_4 = 1 \\ -c_1 + c_2 = 3 \\ c_2 - c_4 = -2 \\ -c_3 = 2 \end{cases}$, and the linear system has the solution $c_1 = -1, c_2 = 2, c_3 = -2$, and $c_4 = 4$, we have that $[\mathbf{v}]_B = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 4 \end{bmatrix}$.

8. Since $c_1 \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$ leads to the linear system

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ -1 & 1 & -1 & 1 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 3 & 3 \end{array}\right], \xrightarrow{\text{which reduces to}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right], \text{ then } [\mathbf{v}]_B = \begin{bmatrix} 2 \\ 2 \\ 1 \\ -1 \end{bmatrix}.$$

9. $[\mathbf{v}]_{B_1} = \begin{bmatrix} -1/4 \\ 1/8 \end{bmatrix}$; $[\mathbf{v}]_{B_2} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$

10. $[\mathbf{v}]_{B_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $[\mathbf{v}]_{B_2} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$

11. $[\mathbf{v}]_{B_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$; $[\mathbf{v}]_{B_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

12. $[\mathbf{v}]_{B_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$; $[\mathbf{v}]_{B_2} = \begin{bmatrix} \frac{1}{3} \\ 1 \\ \frac{7}{3} \\ -\frac{1}{3} \end{bmatrix}$

13. The column vectors for the transition matrix from a basis B_1 to a basis B_2 are the coordinate vectors for the vectors in B_1 relative to B_2. Hence, $[I]_{B_1}^{B_2} = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{B_2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{B_2} \right] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then this matrix transforms coordinates relative to B_1 to coordinates relative to B_2, that is, $[\mathbf{v}]_{B_2} = [I]_{B_1}^{B_2}[\mathbf{v}]_{B_1} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$.

14. $[I]_{B_1}^{B_2} = \begin{bmatrix} 1/3 & 2/3 \\ -7/6 & 1/6 \end{bmatrix}$; $[\mathbf{v}]_{B_2} = [I]_{B_1}^{B_2}[\mathbf{v}]_{B_1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -4/3 \end{bmatrix}$

15. $[I]_{B_1}^{B_2} = \begin{bmatrix} 3 & 2 & 1 \\ -1 & -2/3 & 0 \\ 0 & -1/3 & 0 \end{bmatrix}$; $[\mathbf{v}]_{B_2} = [I]_{B_1}^{B_2}[\mathbf{v}]_{B_1} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

16. $[I]_{B_1}^{B_2} = \begin{bmatrix} -2 & 4/3 & -4/3 \\ 1 & 1/3 & 2/3 \\ 0 & 1/3 & -1/3 \end{bmatrix}$; $[\mathbf{v}]_{B_2} = [I]_{B_1}^{B_2}[\mathbf{v}]_{B_1} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$

17. Notice that the only difference in the bases B_1 and B_2 is the order in which the polynomials $1, x$, and x^2 are given. As a result the column vectors of the transition matrix are the coordinate vectors only permuted. That is, $[I]_{B_1}^{B_2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then $[\mathbf{v}]_{B_2} = [I]_{B_1}^{B_2}[\mathbf{v}]_{B_1} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$.

18. Since $[\mathbf{v}]_{B_2} = [I]_{B_1}^{B_2} = \left[[x^2 - 1]_{B_2} \; [2x^2 + x + 1]_{B_2} \; [-x + 1]_{B_2} \right] = \begin{bmatrix} 1/4 & 5/8 & 3/8 \\ -1 & -1/2 & 1/2 \\ 3/4 & 11/8 & -3/8 \end{bmatrix}$, then $[\mathbf{v}]_{B_2} = [I]_{B_1}^{B_2}[\mathbf{v}]_{B_1} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13/8 \\ -1/2 \\ 11/8 \end{bmatrix}$.

19. Since the equation $c_1\begin{bmatrix}-1\\1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\0\\1\end{bmatrix}+c_3\begin{bmatrix}-1\\1\\0\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}$ gives

$$\left[\begin{array}{ccc|c}-1&1&-1&a\\1&0&1&b\\1&1&0&c\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&0&0&-a-b+c\\0&1&0&a+b\\0&0&1&a+2b-c\end{array}\right],\text{ we have that }\begin{bmatrix}a\\b\\c\end{bmatrix}_B=\begin{bmatrix}-a-b+c\\a+b\\a+2b-c\end{bmatrix}.$$

20. Since

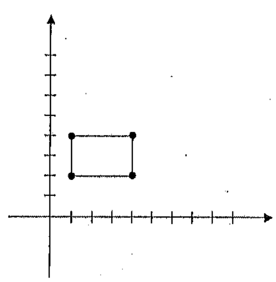
$$\begin{bmatrix}1&0&0&-1&a\\0&-1&-1&0&b\\1&1&-1&0&c\\0&-1&0&-1&d\end{bmatrix}\xrightarrow{\text{reduces to}}\begin{bmatrix}1&0&0&0&2a+b-c-2d\\0&1&0&0&-a-b+c+d\\0&0&1&0&a-c-d\\0&0&0&1&a+b-c-2d\end{bmatrix},\text{ then }\left[\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\right]_B=\begin{bmatrix}2a+b-c-2d\\-a-b+c+d\\a-c-d\\a+b-c-2d\end{bmatrix}$$

21. a. $[I]_{B_1}^{B_2}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$ **b.** $[\mathbf{v}]_{B_2}=[I]_{B_1}^{B_2}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}2\\1\\3\end{bmatrix}$

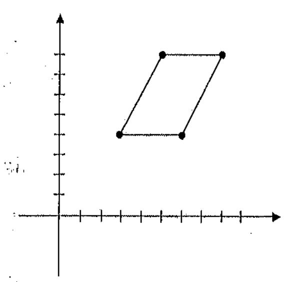
22. a. $[I]_{B_1}^{B_2}=\begin{bmatrix}1&-1\\1&0\end{bmatrix}$ **b.** $[I]_{B_2}^{B_1}=\begin{bmatrix}0&1\\-1&1\end{bmatrix}$ **c.** Since $\begin{bmatrix}1&-1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, then $([I]_{B_1}^{B_2})^{-1}=[I]_{B_2}^{B_1}$.

23. a. $[I]_S^B=\begin{bmatrix}1&1\\0&2\end{bmatrix}$ **b.** $\begin{bmatrix}1\\2\end{bmatrix}_B=\begin{bmatrix}3\\4\end{bmatrix}$; $\begin{bmatrix}1\\4\end{bmatrix}_B=\begin{bmatrix}5\\8\end{bmatrix}$; $\begin{bmatrix}4\\2\end{bmatrix}_B=\begin{bmatrix}6\\4\end{bmatrix}$; $\begin{bmatrix}4\\4\end{bmatrix}_B=\begin{bmatrix}8\\8\end{bmatrix}$

c.

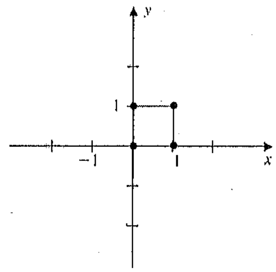


d.

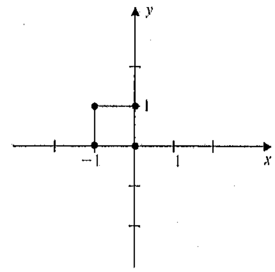


24. a. $[\mathbf{v}]_B=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\cos\theta-y\sin\theta\\x\sin\theta+y\cos\theta\end{bmatrix}$

b.



c. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}_B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



25. a. Since $\mathbf{u}_1 = -\mathbf{v}_1 + 2\mathbf{v}_2, \mathbf{u}_2 = -\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$, and $\mathbf{u}_3 = -\mathbf{v}_2 + \mathbf{v}_3$, the coordinates of $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ relative to B_2 are

$$[\mathbf{u}_1]_{B_2} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, [\mathbf{u}_2]_{B_2} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, [\mathbf{u}_3]_{B_2} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \text{ so } [I]_{B_1}^{B_2} = \begin{bmatrix} -1 & -1 & 0 \\ 2 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

b. $[2\mathbf{u}_1 - 3\mathbf{u}_2 + \mathbf{u}_3]_{B_2} = [I]_{B_1}^{B_2} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}.$

Exercise Set 3.5

1. a. Let $y = e^{rx}$, so that $y' = re^{rx}$ and $y'' = r^2e^{rx}$. Substituting these into the differential equations gives the auxiliary equation $r^2 - 5r + 6 = 0$. Factoring, we have that $(r-3)(r-2) = 0$ and hence, two distinct solutions are $y_1 = e^{2x}$ and $y_2 = e^{3x}$.

b. Since $W[y_1, y_2](x) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x} > 0$ for all x, the two solutions are linear independent. **c.** The general solution is the linear combination $y(x) = C_1e^{2x} + C_2e^{3x}$, where C_1 and C_2 are arbitrary constants.

2. a. Let $y = e^{rx}$, so that $y' = re^{rx}$ and $y'' = r^2e^{rx}$. Then the auxiliary equation is $r^2 + 3r + 2 = 0 = (r+1)(r+2) = 0$ and hence, two distinct solutions are $y_1 = e^{-x}$ and $y_2 = e^{-2x}$.

b. Since $W[y_1, y_2](x) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-x} \neq 0$ for all x, the two solutions are linear independent. **c.** The general solution is the linear combination $y(x) = C_1e^{-x} + C_2e^{-2x}$, where C_1 and C_2 are arbitrary constants.

3. a. Let $y = e^{rx}$, so that $y' = re^{rx}$ and $y'' = r^2e^{rx}$. Substituting these into the differential equation gives the auxiliary equation $r^2 + 4r + 4 = 0$. Factoring, we have that $(r+2)^2 = 0$. Since the auxiliary equation has only one root of multiplicity 2, two distinct solutions are $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$.

b. Since $W[y_1, y_2](x) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix} = e^{-4x} > 0$ for all x, the two solutions are linearly independent. **c.** The general solution is the linear combination $y(x) = C_1e^{-2x} + C_2xe^{-2x}$, where C_1 and C_2 are arbitrary constants.

4. a. Let $y = e^{rx}$, so that $y' = re^{rx}$ and $y'' = r^2e^{rx}$. Then the auxiliary equation is $r^2 - 4r + 5 = 0 = (r+1)(r-5) = 0$ and hence, two distinct solutions are $y_1 = e^{-x}$ and $y_2 = e^{5x}$.

b. Since $W[y_1, y_2](x) = \begin{vmatrix} e^{-x} & e^{5x} \\ -e^{-x} & 5e^{5x} \end{vmatrix} = e^{4x} > 0$ for all x, the two solutions are linear independent. **c.** The general solution is the linear combination $y(x) = C_1e^{-x} + C_2e^{5x}$, where C_1 and C_2 are arbitrary constants.