Exercise Set 3.3

In Section 3.3 of the text, the connection between a spanning set of a vector space and linear independence is completed. The minimal spanning sets, minimal in the sense of the number of vectors in the set, are those that are linear independent. A basis for a vector space V is a set B such that B is linearly independent and $\mathbf{span}(B) = V$. For example,

- $B = \{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ is a basis for $\mathbb{R}^n$
- $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2\times 2}$
- $B = \{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathcal{P}_n$.

Every vector space has infinitely many bases. For example, if $c \neq 0$, then $B = \{c\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ is another basis for $\mathbb{R}^n$. But all bases for a vector space have the same number of vectors, called the dimension of the vector space, and denoted by $\dim(V)$. As a consequence of the bases noted above:

- $\dim(\mathbb{R}^n) = n$
- $\dim(M_{2\times 2}) = 4$ and in general $\dim(M_{m\times n}) = mn$
- $\dim(\mathcal{P}_n) = n + 1$.

If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ is a subset of a vector space V and $\dim(V) = n$ recognizing the following possibilities will be useful in the exercise set:

- If the number of vectors in S exceeds the dimension of V, that is, $m > n$, then S is linearly dependent and hence, can not be a basis.
- If $m > n$, then $\mathbf{span}(S)$ can equal V, but in this case some of the vectors are linear combinations of others and the set S can be trimmed down to a basis for V.
- If $m \leq n$, then the set S can be either linearly independent of linearly dependent.
- If $m < n$, then S can not be a basis for V, since in this case $\mathbf{span}(S) \neq V$.
- If $m < n$ and the vectors in S are linearly independent, then S can be expanded to a basis for V.
- If $m = n$, then S will be a basis for V if either S is linearly independent or $\mathbf{span}(S) = V$. So in this case it is sufficient to verify only one of the conditions.

The two vectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ are linearly independent but can not be a basis for $\mathbb{R}^3$ since all bases for $\mathbb{R}^3$ must have three vectors. To expand to a basis start with the matrix

$$\begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{reduce to the echelon form}} \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}.$$

The pivots in the echelon form matrix are located in columns one, two and four, so the corresponding column vectors in the original matrix form the basis. So

$$B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis for } \mathbb{R}^3.$$

To trim a set of vectors that span the space to a basis the procedure is the same. For example, the set

$$S = \left\{ \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\}$$

is not a basis for $\mathbb{R}^3$ since there are four vectors and hence, S is linearly dependent. Since

$$\begin{bmatrix} 0 & 2 & 0 & 3 \\ -1 & 2 & 2 & -1 \\ -1 & 1 & 2 & -1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} -1 & 2 & 2 & -1 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix},$$

then the span of the four vectors is $\mathbb{R}^3$. The pivots in the reduced matrix are in columns one, two and four, so a basis for $\mathbb{R}^3$ is $\left\{ \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\}$.

Solutions to Exercises

1. Since $\dim(\mathbb{R}^3) = 3$ every basis for $\mathbb{R}^3$ has three vectors. Therefore, since S has only two vectors it is not a basis for $\mathbb{R}^3$.

2. Since $\dim(\mathbb{R}^2) = 2$ every basis for $\mathbb{R}^2$ has two vectors. Therefore, since S has three vectors it is linearly dependent and hence it is not a basis for $\mathbb{R}^2$.

3. Since the third vector can be written as the sum of the first two, the set S is linearly dependent and hence, is not a a basis for $\mathbb{R}^3$.

4. Since $\dim(\mathcal{P}_3) = 4$ every basis has four polynomials and hence S is not a basis.

5. Since the third polynomial is a linear combination of the first two, the set S is linearly dependent and hence is not a basis for $\mathcal{P}_3$.

6. Since the matrix

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

the set S is linearly dependent and hence, is not a basis.

7. The two vectors in S are not scalar multiples and hence, the set S is linearly independent. Since every linearly independent set of two vectors in $\mathbb{R}^2$ is a basis, the set S is a basis.

8. Since $\det\left(\begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}\right) = -2$, the set S is linearly independent and hence, is a basis for $\mathbb{R}^2$.

9. Since

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & -2 & 2 \\ 1 & -3 & -2 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -5 \end{bmatrix},$$

S is a linearly independent set of three vectors in $\mathbb{R}^3$ and hence, S is a basis.

10. Since

$$\begin{bmatrix} -1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & -3 & 2 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} -1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

S is a linearly independent set of three vectors in $\mathbb{R}^3$ and hence, S is a basis.

11. Since

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix},$$

the set S is a linearly independent set of four matrices in $M_{2\times 2}$. Since $\dim(M_{2\times 2}) = 4$, then S is a basis.

12. Since

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

S is a linearly independent set of three polynomials in $\mathcal{P}_2$ and hence, is a basis.

13. Since

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix},$$

the set S is a linearly independent set of three vectors in $\mathbb{R}^3$, so is a basis.

14. Since

$$\begin{bmatrix} 2 & 5 & 3 \\ -2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 2 & 5 & 3 \\ 0 & 6 & 4 \\ 0 & 0 & 1 \end{bmatrix},$$

the set S is a linearly independent set of three vectors in $\mathbb{R}^3$, so is a basis.

15. Since

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 1 & 1 & 4 & 2 \\ -1 & 3 & 2 & 0 \\ 1 & 1 & 5 & 3 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the homogeneous linear system has infinitely many solutions so the set S is linearly dependent and is therefore not a basis for $\mathbb{R}^4$.

16. Since

$$\begin{bmatrix} -1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 2 & -1 & 2 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} -1 & 2 & 1 & 2 \\ 0 & 3 & 4 & 3 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 0 & 32 \end{bmatrix},$$

the set S is a linearly independent set of four vectors in $\mathbb{R}^4$, so is a basis.

17. Notice that $\frac{1}{3}(2x^2 + x + 2 + 2(-x^2 + x) - 2(1)) = x$ and $2x^2 + x + 2 + (-x^2 + x) - 2x - 2 = x^2$, so the span of S is $\mathcal{P}_2$. Since $\dim(\mathcal{P}_2) = 3$, the set S is a basis.

18. Since

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

the set S is a linearly independent set of four matrices in $M_{2\times 2}$, so is a basis.

19. Every vector in S can be written as $\begin{bmatrix} s+2t \\ -s+t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Since the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ are linear independent a basis for S is $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $\dim(S) = 2$.

20. Since every matrix in S can be written in the form

$$\begin{bmatrix} a & a+d \\ a+d & d \end{bmatrix} = a\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

and the two matrices on the right hand side are linearly independent, a basis for S is $B = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$. Consequently, $\dim(S) = 2$.

21. Every 2×2 symmetric matrix has the form $\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Since the three matrices on the right are linearly independent a basis for S is $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and $\dim(S) = 3$.

22. Every 2×2 skew symmetric matrix has the form $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and hence, a basis for S is $B = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ with $\dim(S) = 1$.

23. Since every polynomial $p(x)$ in S satisfies $p(0) = 0$, we have that $p(x) = ax+bx^2$. Therefore, a basis for S is $B = \{x, x^2\}$ and $\dim(S) = 2$.

24. If $p(x) = ax^3+bx^2+cx+d$ and $p(0) = 0$, then $d = 0$, so $p(x)$ has the form $p(x) = ax^2 + bx + cx$. If in addition, $p(1) = 0$, then $a + b + c = 0$, so $c = -a - b$ and hence
$p(x) = ax^3+bx^2+(-a-b)x = a(x^3-x)+b(x^2-x)$. Since $x^3 - x$ and $x^2 - x$ are linear independent, then a basis for S is $\{x^3-x, x^2-x\}$, so $\dim(S) = 2$.

25. Since $\det\left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 2 \\ -1 & -2 & 1 \end{bmatrix}\right) = -4$, the set S is already a basis for $\mathbb{R}^3$ since it is a linearly independent set of three vectors in $\mathbb{R}^3$.

26. Since $\begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$, the vectors in S are linearly dependent. Since the first two vectors are not scalar multiples of each other a basis for **span**(S) is $\left\{\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}\right\}$.

27. The vectors can not be a basis since a set of four vectors in $\mathbb{R}^3$ is linearly dependent. To trim the set down to a basis for the span row reduce the matrix with column vectors the vectors in S. This gives

$$\begin{bmatrix} 2 & 0 & -1 & 2 \\ -3 & 2 & -1 & 3 \\ 0 & 2 & 0 & -1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 2 & -\frac{5}{2} & 6 \\ 0 & 0 & \frac{5}{2} & -7 \end{bmatrix}.$$

A basis for the span consists of the column vectors in the original matrix corresponding to the pivot columns of the row echelon matrix. So a basis for the span of S is given by
$B = \left\{\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}\right\}$. Observe that **span**$(S) = \mathbb{R}^3$.

28. The vectors can not be a basis since a set of four vectors in $\mathbb{R}^3$ is linearly dependent. To trim the set down to a basis for the span row reduce the matrix with column vectors the vectors in S. This gives

$$\begin{bmatrix} -2 & 1 & -3 & 1 \\ 0 & 0 & -3 & 2 \\ 2 & -3 & -2 & -2 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & -2 & -5 & -1 \\ 0 & 0 & -3 & 2 \end{bmatrix}.$$

A basis for the span consists of the column vectors in the original matrix corresponding to the pivot columns of the row echelon matrix. So a basis for the span of S is given by
$B = \left\{\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ -2 \end{bmatrix}\right\}$. Observe that **span**$(S) = \mathbb{R}^3$.

29. The vectors can not be a basis since a set of four vectors in $\mathbb{R}^3$ is linearly dependent. To trim the set down to a basis for the span row reduce the matrix with column vectors the vectors in S. This gives

$$\begin{bmatrix} 2 & 0 & 2 & 4 \\ -3 & 2 & -1 & 0 \\ 0 & 2 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & 2 & 4 \\ 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

A basis for the span consists of the column vectors in the original matrix corresponding to the pivot columns of the row echelon matrix. So a basis for the span of S is given by

$B = \left\{ \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} \right\}$. Observe that $\mathbf{span}(S) = \mathbb{R}^3$.

30. The vectors can not be a basis since a set of four vectors in $\mathbb{R}^3$ is linearly dependent. To trim the set down to a basis for the span row reduce the matrix with column vectors the vectors in S. This gives

$$\begin{bmatrix} 2 & 1 & 0 & 2 \\ 2 & -1 & 2 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & -2 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

A basis for the span consists of the column vectors in the original matrix corresponding to the pivot columns of the row echelon matrix. So a basis for the span of S is given by

$B = \left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$. Observe that $\mathbf{span}(S) = \mathbb{R}^3$.

31. Form the 3×5 matrix with first two column vectors the vectors in S and then augment the identity matrix. Reducing this matrix, we have that

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & -2 & -1 & 1 \end{bmatrix}.$$

A basis for $\mathbb{R}^3$ consists of the column vectors in the original matrix corresponding to the pivot columns of the row echelon matrix. So a basis for $\mathbb{R}^3$ containing S is $B = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

32. Form the 3×5 matrix with first two column vectors the vectors in S and then augment the identity matrix. Reducing this matrix, we have that

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}.$$

A basis for $\mathbb{R}^3$ consists of the column vectors in the original matrix corresponding to the pivot columns of the row echelon matrix. So a basis for $\mathbb{R}^3$ containing S is $B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

33. A basis for $\mathbb{R}^4$ containing S is

$$B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

34. A basis for $\mathbb{R}^4$ containing S is

$$B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

35. A basis for $\mathbb{R}^3$ containing S is

$$B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

36. A basis for $\mathbb{R}^3$ containing S is

$$B = \left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

37. Let $\mathbf{e}_{ii}$ denote the $n \times n$ matrix with a 1 in the row i, column i component and 0 in all other locations. Then $B = \{\mathbf{e}_{ii} \mid 1 \le i \le n\}$ is a basis for the subspace of all $n \times n$ diagonal matrices.

38. Consider the equation $c_1(c\mathbf{v_1}) + c_2(c\mathbf{v_2}) + \cdots + c_n(c\mathbf{v_n}) = \mathbf{0}$. Then $c(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}) = \mathbf{0}$ and since $c \neq 0$, we have that $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{0}$. Now since S is a basis, it is linearly independent, so $c_1 = c_2 = \cdots = c_n = 0$ and hence S' is a basis.

39. It is sufficient to show that the set S' is linearly independent. Consider the equation $c_1A\mathbf{v_1} + c_2A\mathbf{v_2} + \cdots + c_nA\mathbf{v_n} = \mathbf{0}$. This is equivalent to $A(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}) = \mathbf{0}$. Multiplying both sides of this equation by A^{-1} gives the equation $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{0}$. Since S is linearly independent, then $c_1 = c_2 = \cdots c_n = 0$, so that S' is linearly independent.

40. To solve the homogeneous equation $A\mathbf{x} = \mathbf{0}$ consider the matrix

$$\begin{bmatrix} 3 & 3 & 1 & 3 \\ -1 & 0 & -1 & -1 \\ 2 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{\text{that reduces to}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So a vector is a solution provided it has the form $\mathbf{x} = \begin{bmatrix} -z \\ \frac{2}{3}z \\ z \\ 0 \end{bmatrix} = z\begin{bmatrix} -1 \\ \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$. Hence, a basis for S is $\left\{\begin{bmatrix} -1 \\ \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}\right\}$.

41. Since H is a subspace of V, then $H \subseteq V$. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be a basis for H, so that S is a linearly independent set of vectors in V. Since $\dim(V) = n$, then S is also a basis for V. Now let $\mathbf{v}$ be a vector in V. Since S is a basis for V, then there exist scalars $c_1, c_2, \ldots c_n$ such that $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{v}$. Since S is a basis for H, then $\mathbf{v}$ is a linear combination of vectors in H and is therefore, also in H. Hence, $V \subseteq H$ and we have that $H = V$.

42. Since $S = \{ax^3 + bx^2 + cx \mid a, b, c \in \mathbb{R}\}$, then $\dim(S) = 3$. A polynomial $q(x) = ax^3 + bx^2 + cx + d$ is in T if and only if $a + b + c + d = 0$, that is $d = -a - b - c$. Hence, a polynomial in T has the form $q(x) = a(x^3 - 1) + b(x^2 - 1) + c(x - 1)$, so that $\dim(T) = 3$. Now $S \cap T = \{ax^3 + bx^2 + cx \mid a + b + c = 0\}$. Hence, a polynomial $q(x)$ is in $S \cap T$ if and only if $q(x) = a(x^3 - x) + b(x^2 - x)$ and hence, $\dim(S \cap T) = 2$.

43. Every vector in W can be written as a linear combination of the form

$$\begin{bmatrix} 2s + t + 3r \\ 3s - t + 2r \\ s + t + 2r \end{bmatrix} = s\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + r\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}.$$

But $\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and hence, $\textbf{span}\left\{\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\} = \textbf{span}\left\{\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}\right\}$.

Since $B = \left\{\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$ is linear independent, B is a basis for W, so that $\dim(W) = 2$.

44. Since $S = \left\{ s\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \middle| \; s, t \in \mathbb{R} \right\}$, then $\dim(S) = 2$. Since

$T = \left\{ s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \middle| \; s, t \in \mathbb{R} \right\}$, we also have that $\dim(T) = 2$. For the intersection, since