**22. a.** Since

$$\begin{bmatrix} t \\ 1+t \end{bmatrix} = \begin{bmatrix} t \\ 1+t \end{bmatrix} + \begin{bmatrix} s \\ 1+s \end{bmatrix} = \begin{bmatrix} t+s \\ 1+(t+s) \end{bmatrix} \Leftrightarrow s = 0,$$

the additive identity is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. **b.** Since the other nine vector space properties also hold, $V$ is a vector space.

**c.** Since $0 \odot \begin{bmatrix} t \\ 1+t \end{bmatrix} = \begin{bmatrix} 0t \\ 1+0t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the additive identity, then $0 \odot \mathbf{v} = \mathbf{0}$.

**23. a.** The additive identity is $\mathbf{0} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Let $\mathbf{u} = \begin{bmatrix} 1+a \\ 2-a \\ 3+2a \end{bmatrix}$. Then the additive inverse is $-\mathbf{u} = \begin{bmatrix} 1-a \\ 2+a \\ 3-2a \end{bmatrix}$. **b.** Each of the ten vector space axioms is satisfied. **c.** $0 \odot \begin{bmatrix} 1+t \\ 2-t \\ 3+2t \end{bmatrix} = \begin{bmatrix} 1+0t \\ 2-0t \\ 3+2(0)t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

**24.** Since $S$ is a subset of $\mathbb{R}^3$ with the same standard operations only vector space axioms (1) and (6) need to be verified since the others are inherited from the vector space $\mathbb{R}^3$. If $\mathbf{w}_1$ and $\mathbf{w}_2$ are in $S$, then let $\mathbf{w}_1 = a\mathbf{u}+b\mathbf{v}$ and $\mathbf{w}_2 = c\mathbf{u}+d\mathbf{v}$. Then $\mathbf{w}_1 + \mathbf{w}_2 = (a+c)\mathbf{u} + (b+d)\mathbf{v}$ and $k(a\mathbf{u}+b\mathbf{v}) = (ka)\mathbf{u} + (kb)\mathbf{v}$ are also in $S$.

**25.** Each of the ten vector space axioms is satisfied.

**26.** The set $S$ is a plane through the origin in $\mathbb{R}^3$, so the sum of vectors in $S$ remains in $S$ and a scalar times a vector in $S$ remains in $S$. Since the other vector space properties are inherited form the vector space $\mathbb{R}^3$, then $S$ is a vector space.

**27.** Each of the ten vector space axioms is satisfied.

**28. a.** Since $\cos(0) = 1$ and $\sin(0) = 0$, the additive identity is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The additive inverse of $\begin{bmatrix} \cos t_1 \\ \sin t_1 \end{bmatrix}$ is $\begin{bmatrix} \cos(-t_1) \\ \sin(-t_1) \end{bmatrix} = \begin{bmatrix} \cos t_1 \\ -\sin t_1 \end{bmatrix}$. **b.** The ten required properties hold making $V$ a vector space. **c.** The additive identity in this case is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since $\cos t$ and $\sin t$ are not both 0 for any value of $t$, then $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in $V$, so $V$ is not a vector space.

**29.** Since $(f+g)(0) = f(0) + g(0) = 1 + 1 = 2$, then $V$ is not closed under addition and hence is not a vector space.

**30.** Since $c \odot (d \odot f)(x) = c \odot (f(x+d)) = f(x+c+d)$ and $(cd) \odot f(x) = f(x+cd)$, do not agree for all scalars, $V$ is not a vector space.

**31. a.** The zero vector is given by $f(x+0) = x^3$ and $-f(x+t) = f(x-t)$. **b.** Each of the ten vector space axioms is satisfied.

## Exercise Set 3.2

A subset $W$ of a vector space $V$ is a subspace of the vector space if vectors in $W$, using the same addition and scalar multiplication of $V$, satisfy the ten vector space properties. That is, $W$ is a vector space. Many of the vector space properties are inherited by $W$ from $V$. For example, if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $W$, then they are also vectors in $V$, so that $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$. On the other hand, the additive identity may not be a vector in $W$, which is a requirement for being a vector space. To show that a subset is a subspace it is sufficient to verify that

if $\mathbf{u}$ and $\mathbf{v}$ are in $W$ and $c$ is a scalar, then $\mathbf{u} + c\mathbf{v}$ is another vector in $W$.

For example, let

$$W = \left\{ \left. \begin{bmatrix} s-2t \\ t \\ s+t \end{bmatrix} \right| s,t \in \mathbb{R} \right\},$$

which is a subset of $\mathbb{R}^3$. Notice that if $s = t = 0$, then the additive identity $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for the vector space $\mathbb{R}^3$ is also in $W$. Let $\mathbf{u} = \begin{bmatrix} s-2t \\ t \\ s+t \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} a-2b \\ b \\ a+b \end{bmatrix}$ denote two arbitrary vectors in $W$. Notice that we have to use different parameters for the two vectors since the vectors may be different. Next we form the linear combination

$$\mathbf{u} + c\mathbf{v} = \begin{bmatrix} s-2t \\ t \\ s+t \end{bmatrix} + c \begin{bmatrix} a-2b \\ b \\ a+b \end{bmatrix}$$

and simplify the sum to one vector. So

$$\mathbf{u} + c\mathbf{v} = \begin{bmatrix} s-2t \\ t \\ s+t \end{bmatrix} + c \begin{bmatrix} a-2b \\ b \\ a+b \end{bmatrix} = \begin{bmatrix} (s-2t)+(a-2b) \\ t+b \\ (s+t)+(a+b) \end{bmatrix}$$

but this is not sufficient to show the vector is in $W$ since the vector must be written in terms of just two parameters in the locations described in the definition of $W$. Continuing the simplification, we have that

$$\mathbf{u} + c\mathbf{v} = \begin{bmatrix} (s+a)-2(t+b) \\ t+b \\ (s+a)+(t+b) \end{bmatrix}$$

and now the vector $\mathbf{u}+c\mathbf{v}$ is in the required form with two parameters $s+a$ and $t+b$. Hence, $W$ is a subspace of $\mathbb{R}^3$. An arbitrary vector in $W$ can also be written as

$$\begin{bmatrix} s-2t \\ t \\ s+t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

and in this case $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ are linearly independent, so $W$ is a plane in $\mathbb{R}^3$. The set $W$ consists of all linear combinations of the two vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, called the span of the vectors and written

$$W = \textbf{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

The span of a set of vectors is always a subspace. Important facts to keep in mind are:

- There are linearly independent vectors that span a vector space. The coordinate vectors of $\mathbb{R}^3$ are a simple example.
- Two linearly independent vectors can not span $\mathbb{R}^3$, since they describe a plane and one vector can not span $\mathbb{R}^2$ since all linear combinations describe a line.

- Two linearly dependent vectors can not span $\mathbb{R}^2$. Let $S = \textbf{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \end{bmatrix}\right\}$. If $\mathbf{v}$ in in $S$, then there are scalars $c_1$ and $c_2$ such that
$$\mathbf{v} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -4 \\ -2 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\left(-2\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = (c_1 - 2c_2)\begin{bmatrix} 2 \\ 1 \end{bmatrix},$$
and hence, every vector in the span of $S$ is a linear combination of only one vector.
- A linearly dependent set of vectors can span a vector space. For example, let $S = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$. Since the coordinate vectors are in $S$, then $\textbf{span}(S) = \mathbb{R}^2$, but the vectors are linearly dependent since $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

In general, to determine whether or not a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is in $\textbf{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, start with the vector equation
$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{v},$$
and then solve the resulting linear system. These ideas apply to all vector spaces not just the Euclidean spaces. For example, if $S = \{A \in M_{2\times 2} | A \text{ is invertible}\}$, then $S$ is not a subspace of the vector space of all $2 \times 2$ matrices. For example, the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are both invertible, so are in $S$, but $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, which is not invertible. To determine whether of not $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ is in the span of the two matrices $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$, start with the equation
$$c_1\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} c_1 - c_2 & 2c_1 \\ c_2 & c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$
The resulting linear system is $c_1 - c_2 = 2, 2c_1 = -1, c_2 = 1, c_1 + c_2 = 1$, is inconsistent and hence, the matrix is not in the span of the other two matrices.

## Solutions to Exercises

**1.** Let $\begin{bmatrix} 0 \\ y_1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ y_2 \end{bmatrix}$ be two vectors in $S$ and $c$ a scalar. Then $\begin{bmatrix} 0 \\ y_1 \end{bmatrix} + c\begin{bmatrix} 0 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ y_1 + cy_2 \end{bmatrix}$ is in $S$, so $S$ is a subspace of $\mathbb{R}^2$.

**2.** The set $S$ is not a subspace of $\mathbb{R}^2$. If $\mathbf{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \notin S$.

**3.** The set $S$ is not a subspace of $\mathbb{R}^2$. If $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \notin S$.

**4.** The set $S$ is not a subspace of $\mathbb{R}^2$. If $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$, then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix} \notin S$ since $(3/2)^2 + 0^2 = 9/4 > 1$.

**5.** The set $S$ is not a subspace of $\mathbb{R}^2$. If $\mathbf{u} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $c = 0$, then $c\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin S$.

**6.** The set $S$ is a subspace since $\begin{bmatrix} x \\ 3x \end{bmatrix} + c\begin{bmatrix} y \\ 3y \end{bmatrix} = \begin{bmatrix} x + cy \\ 3(x + cy) \end{bmatrix} \in S$.

**7.** Since

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + c \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + cy_1 \\ x_2 + cy_2 \\ x_3 + cy_3 \end{bmatrix}$$

and $(x_1 + cy_1) + (x_3 + cy_3) = (x_1 + x_3) + c(y_1 + y_3) = -2(c+1) = 2$ if and only if $c = -2$, $S$ is not a subspace of $\mathbb{R}^3$.

**8.** Suppose $x_1x_2x_3 = 0$ and $y_1y_2y_3 = 0$, where $x_1 = 0, y_3 = 0$, and all other components are nonzero. Then $(x_1 + y_1)(x_2 + y_2)(x_3 + y_3) \neq 0$, so $S$ is not a subspace.

**9.** Since for all real numbers $s, t, c$, we have that

$$\begin{bmatrix} s - 2t \\ s \\ t + s \end{bmatrix} + c \begin{bmatrix} x - 2y \\ x \\ y + x \end{bmatrix} = \begin{bmatrix} (s + cx) - 2(t + cy) \\ s + cx \\ (t + cy) + (s + cx) \end{bmatrix},$$

is in $S$, then $S$ is a subspace.

**10.** For any two vectors in $S$, we have $\begin{bmatrix} x_1 \\ 2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ 2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ 4 \\ x_2 + y_2 \end{bmatrix}$, which is not in $S$ and hence, $S$ is not a subspace.

**11.** If $A$ and $B$ are symmetric matrices and $c$ is a scalar, then $(A + cB)^t = A^t + cB^t = A + cB$, so $S$ is a subspace.

**12.** If $A$ and $B$ are idempotent matrices, then $(A + B)^2 = A^2 + AB + BA + B^2$ will equal $A^2 + B^2 = A + B$ if and only if $AB = -BA$, so $S$ is not a subspace.

**13.** Since the sum of invertible matrices may not be invertible, $S$ is not a subspace.

**14.** If $A$ and $B$ are skew symmetric matrices and $c$ is a scalar, then $(A + cB)^t = A^t + cB^t = -A + c(-B) = -(A + cB)$, so $S$ is a subspace.

**15.** If $A$ and $B$ are upper triangular matrices, then $A + B$ and $cB$ are also upper triangular, so $S$ is a subspace.

**16.** If $A$ and $B$ are diagonal matrices and $c$ is a scalar, then $A + cB$ is a diagonal matrix and hence, $S$ is a subspace.

**17.** The set $S$ is a subspace.

**18.** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ such that $a + c = 0$ and $e + g = 0$, then $A + B = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$ with

$$(a + e) + (d + h) = (a + d) + (e + h) = 0$$

and hence, $S$ is a subspace.

**19.** The set $S$ is not a subspace since $x^3 - x^3 = 0$, which is not a polynomial of degree 3.

**20.** The set $S$ is not a subspace since $(x^2 + x) - x^2 = x$, which is not a polynomial of even degree.

**21.** If $p(x)$ and $q(x)$ are polynomials with $p(0) = 0$ and $q(0) = 0$, then

$$(p + q)(0) = p(0) + q(0) = 0,$$

and

$$(cq)(0) = cq(0) = 0,$$

so $S$ is a subspace.

**22.** Yes, since $ax^2 + c(bx^2) = (a + cb)x^2 \in S$.

**23.** The set $S$ is not a subspace, since for example $(2x^2 + 1) - (x^2 + 1) = x^2$, which is not in $S$.

**24.** The set $S$ is a subspace (assuming the zero polynomial is in the set).

**25.** The vector $\mathbf{v}$ is in the span of $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ provided there are scalars $c_1, c_2$, and $c_3$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Row reduce the augmented matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3 \mid \mathbf{v}]$. We have that

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & -2 \end{array}\right],$$

and since the linear system has a (unique) solution, the vector $\mathbf{v}$ is in the span.

**26.** Since

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 1 & -1 & 2 & 7 \\ 0 & 1 & 0 & -3 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 3 & 9 \end{array}\right],$$

the linear system has a (unique) solution and hence, the vector $\mathbf{v}$ is in the span.

**27.** Since

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 1 & 1 & -1 & 1 \\ 0 & 2 & -4 & 6 \\ -1 & 1 & -3 & 5 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \text{the matrix } M \text{ is in the span.}$$

**28.** Since

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 2 & -4 & 2 \\ -1 & 1 & -3 & -3 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

the linear system is inconsistent and hence, the matrix $M$ is not in the span.

**29.** Since

$$c_1(1+x) + c_2(x^2-2) + c_3(3x) = 2x^2 - 6x - 11 \text{ if and only if } (c_1 - 2c_2) + (c_1 + 3c_3)x + c_2x^2 = 2x^2 - 6x - 11,$$

we have that $c_1 = -7, c_2 = 2, c_3 = \frac{1}{3}$ and hence, the polynomial is in the span.

**30.** Since

$$c_1(1+x) + c_2(x^2-2) + c_3(3x) = 3x^2 - x - 4 \text{ if and only if } (c_1 - 2c_2) + (c_1 + 3c_3)x + c_2x^2 = 3x^2 - x - 4,$$

we have that $c_1 = 2, c_2 = 3, c_3 = -1$ and hence, the polynomial is in the span.

**31.** Since

$$\left[\begin{array}{cc|c} 2 & 1 & a \\ -1 & 3 & b \\ -2 & -1 & c \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{cc|c} -1 & 3 & b \\ 0 & 7 & a+2b \\ 0 & 0 & a+c \end{array}\right], \text{ then } \mathbf{span}(S) = \left\{ \left[\begin{array}{c} a \\ b \\ c \end{array}\right] \middle| \, a + c = 0 \right\}.$$

**32.** Since

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 1 & 3 & 2 & b \\ 2 & 1 & -1 & c \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 1 & -a+b \\ 0 & 0 & 0 & -5a+3b+c \end{array}\right], \text{ then } \mathbf{span}(S) = \left\{ \left[\begin{array}{c} a \\ b \\ c \end{array}\right] \middle| \, -5a + 3b + c = 0 \right\}.$$

**33.** The equation $c_1 \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, leads to the linear system $c_1 + c_2 = a, 2c_1 - c_2 = b, c_1 = c, c_2 = d$, which gives $\mathbf{span}(S) = \left\{ \begin{bmatrix} a & b \\ \frac{a+b}{3} & \frac{2a-b}{3} \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$.

**34.**

$$\mathbf{span}(S) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \ b + d = 0 \right\}.$$

**35.**

$$\mathbf{span}(S) = \left\{ ax^2 + bx + c \middle| \ a - c = 0 \right\}.$$

**36.** Since

$$\left[\begin{array}{ccc|c} -4 & 2 & 2 & a \\ 0 & -1 & 1 & b \\ 1 & 0 & 1 & c \end{array}\right]$$

$$\xrightarrow{\text{reduces to}} \left[\begin{array}{ccc|c} -4 & 2 & 2 & a \\ 0 & -1 & 1 & b \\ 0 & 0 & 2 & c + \frac{1}{4}a + \frac{1}{2}b \end{array}\right],$$

the span is all polynomials of degree two or less.

**37. a.**

$$\mathbf{span}(S) = \left\{ \begin{bmatrix} a \\ b \\ \frac{b-2a}{3} \end{bmatrix} \middle| \ a, b \in \mathbb{R} \right\}.$$

**b.** The set $S$ is linearly independent.

**38. a.**

$$\mathbf{span}(S) = \left\{ \begin{bmatrix} a \\ b \\ 3a - b \end{bmatrix} \middle| \ a, b \in \mathbb{R} \right\}.$$

**b.** Since $2\mathbf{v_1} - \mathbf{v_2} = \mathbf{v_3}$, the set $S$ is linearly dependent.

**39. a.** $\mathbf{span}(S) = \mathbb{R}^3$.
**b.** The set $S$ is linearly independent.

**40. a.** Since

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & a \\ 2 & 0 & 1 & 1 & b \\ 1 & 3 & 1 & 1 & c \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & a \\ 0 & 2 & 1 & -3 & -2a + b \\ 0 & 0 & -1 & 5 & 3a - 2b + c \end{array}\right],$$

every vector in $\mathbb{R}^3$ is a linear combination of the vectors in $S$ and hence, $\mathbf{span}(S) = \mathbb{R}^3$. **b.** The set $S$ is linearly dependent since there are four vectors in $\mathbb{R}^3$.

**41. a.** $\mathbf{span}(S) = \mathbb{R}^3$ **b.** The set $S$ is linearly dependent. **c.** The set $T$ is also linearly dependent and $\mathbf{span}(T) = \mathbb{R}^3$. **d.** The set $H$ is linearly independent and we still have $\mathbf{span}(H) = \mathbb{R}^3$.

**42. a.** $\mathbf{span}(S) = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \middle| \ a, b, c \in \mathbb{R} \right\}$ **b.** The set $S$ is linearly independent.
**c.** The set $T$ is also linearly independent and $\mathbf{span}(T) = M_{2\times 2}$.

**43. a.** $\mathbf{span}(S) = \mathcal{P}_2$ **b.** The set $S$ is linearly dependent. **c.** $2x^2 + 3x + 5 = 2(1) - (x - 3) + 2(x^2 + 2x)$ **d.** The set $T$ is linearly independent and $\mathbf{span}(T) = \mathcal{P}_3$.

**44. a.** Since

$$\begin{bmatrix} 2s_1 - t_1 \\ s_1 \\ t_1 \\ -s_1 \end{bmatrix} + c \begin{bmatrix} 2s_2 - t_2 \\ s_2 \\ t_2 \\ -s_2 \end{bmatrix} = \begin{bmatrix} 2(s_1 + cs_2) - (t_1 + ct_2) \\ s_1 + cs_2 \\ t_1 + ct_2 \\ -(s_1 + cs_2) \end{bmatrix} \in S,$$

then $S$ is a subspace. **b.** Since $\begin{bmatrix} 2s - t \\ s \\ t \\ -s \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, then $S = \mathbf{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

**c.** Since the vectors $\begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ are not multiples of each other they are linearly independent.

**d.** $S \subsetneq \mathbb{R}^4$

**45. a., b.** Since

$$\begin{bmatrix} -s \\ s-5t \\ 2s+3t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix}, \text{ then } S = \mathbf{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix}\right\}.$$

Therefore, $S$ is a subspace. **c.** The vectors found in part (b) are linearly independent. **d.** Since the span of two linearly independent vectors in $\mathbb{R}^3$ is a plane, then $S \neq \mathbb{R}^3$

**46. a.** The subspace $S$ consists of all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $-a-2b+c+d=0$. **b.** From part (a) not all matrices can be written as a linear combination of the matrices in $S$ and hence, the span of $S$ is not equal to $M_{2\times2}$. **c.** The matrices that generate the set $S$ are linearly independent.

**47.** Since $A(\mathbf{x}+c\mathbf{y}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ if and only if $c=0$, then $S$ is not a subspace.

**48.** If $\mathbf{u}$ and $\mathbf{v}$ are in $S$, then $A(\mathbf{u}+c\mathbf{v}) = A\mathbf{u}+cA\mathbf{v} = \mathbf{0}+\mathbf{0} = \mathbf{0}$ and hence $S$ is a subspace.

**49.** Let $B_1, B_2 \in S$. Since $A$ commutes with $B_1$ and $B_2$, we have that

$$A(B_1+cB_2) = AB_1 + cAB_2 = B_1A + c(B_2A) = (B_1+cB_2)A$$

and hence, $B_1+cB_2 \in S$ and $S$ is a subspace.

**50.** Let $\mathbf{w_1} = \mathbf{u_1}+\mathbf{v_1}$ and $\mathbf{w_2} = \mathbf{u_2}+\mathbf{v_2}$ be two elements of $S$. Let $c$ be a scalar. Then

$$\mathbf{w_1}+c\mathbf{w_2} = \mathbf{u_1}+\mathbf{v_1}+c(\mathbf{u_2}+\mathbf{v_2}) = (\mathbf{u_1}+c\mathbf{u_2})+(\mathbf{v_1}+c\mathbf{v_2}).$$

Since $S$ and $T$ are subspaces, then $\mathbf{u_1}+c\mathbf{u_2} \in S$ and $\mathbf{v_1}+c\mathbf{v_2} \in T$. Therefore, $\mathbf{w_1}+c\mathbf{w_2} \in S+T$ and hence, $S+T$ is a subspace.

**51.** Let $\mathbf{w} \in S+T$, so that $\mathbf{w} = \mathbf{u}+\mathbf{v}$, where $\mathbf{u} \in S$, and $\mathbf{v} \in T$. Then there are scalars $c_1, \ldots, c_m$ and $d_1, \ldots, d_n$ such that $\mathbf{w} = \sum_{i=1}^m c_i\mathbf{u_i} + \sum_{i=1}^n d_i\mathbf{v_i}$. Therefore, $\mathbf{w} \in \mathbf{span}\{\mathbf{u_1}, \ldots, \mathbf{u_m}, \mathbf{v_1}, \ldots, \mathbf{v_n}\}$ and we have shown that $S+T \subseteq \mathbf{span}\{\mathbf{u_1}, \ldots, \mathbf{u_m}, \mathbf{v_1}, \ldots, \mathbf{v_n}\}$.

Now let $\mathbf{w} \in \mathbf{span}\{\mathbf{u_1}, \ldots, \mathbf{u_m}, \mathbf{v_1}, \ldots, \mathbf{v_n}\}$, so there are scalars $c_1, \ldots, c_{m+n}$ such that $\mathbf{w} = c_1\mathbf{u_1}+\cdots+c_m\mathbf{u_m}+c_{m+1}\mathbf{v_1}+\cdots+c_{m+n}\mathbf{v_n}$, which is in $S+T$. Therefore, $\mathbf{span}\{\mathbf{u_1}, \ldots, \mathbf{u_m}, \mathbf{v_1}, \ldots, \mathbf{v_n}\} \subseteq S+T$.

**52. a.** Since

$$\begin{bmatrix} a & -a \\ b & c \end{bmatrix} + k\begin{bmatrix} d & -d \\ e & f \end{bmatrix} = \begin{bmatrix} a+kd & -a-kd \\ b+e & c+f \end{bmatrix} = \begin{bmatrix} a+kd & -(a+kd) \\ b+e & c+f \end{bmatrix}$$

is in $S$, then $S$ is a subspace. Similarly, $T$ is a subspace. **b.** The sets $S$ and $T$ are given by

$$S = \mathbf{span}\left\{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}, T = \mathbf{span}\left\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\},$$

so

$$S+T = \mathbf{span}\left\{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}.$$

But $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so

$$S+T = \mathbf{span}\left\{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\} = M_{2\times2}.$$