and since $(x_1+x_2)-(y_1+y_2)+(z_1+z_2)=(x_1-y_1+z_1)+(x_2-y_2+z_2)=0+0=0$, the sum is also in V. Similarly, $c\mathbf{u}$ is in V for all scalars c. These are the only properties that need to be verified to show that V is a vector space.

Solutions to Exercises

1. In order to show that a set V with an addition and scalar multiplication defined is a vector space all ten properties in Definition 1 must be satisfied. To show that V is not a vector space it is sufficient to show any one of the properties does not hold. Since

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1-x_2 \\ y_1-y_2 \\ z_1-z_2 \end{bmatrix}$$

and

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \oplus \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_2-x_1 \\ y_2-y_1 \\ z_2-z_1 \end{bmatrix}$$

do not agree for all pairs of vectors, the operation $\oplus$ is not commutative, so V is not a vector space.

2. Notice that since scalar multiplication is defined in the standard way the necessary properties hold for this operation. Since the addition of two vectors is another vector, V is closed under addition. Consider

$$c\odot\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = c\odot\left(\begin{bmatrix} x_1+y_1-1 \\ x_2+y_2-1 \\ x_3+y_3-1 \end{bmatrix}\right) = \begin{bmatrix} cx_1+cy_1-c \\ cx_2+cy_2-c \\ cx_3+cy_3-c \end{bmatrix}$$

but

$$c\odot\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \oplus c\odot\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix} \oplus \begin{bmatrix} cy_1 \\ cy_2 \\ cy_3 \end{bmatrix} = \begin{bmatrix} cx_1+cy_1-1 \\ cx_2+cy_2-1 \\ cx_3+cy_3-1 \end{bmatrix},$$

so the expressions $c\odot(\mathbf{u}\oplus\mathbf{v})$ and $c\odot\mathbf{u}\oplus c\odot\mathbf{v}$ do not agree for all pairs of vectors and scalars c. Therefore, V is not a vector space.

3. The operation $\oplus$ is not associative so V is not a vector space. That is,

$$\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) \oplus \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4x_1+4x_2+2x_3 \\ 4y_1+4y_2+2y_3 \\ 4z_1+4z_2+2z_3 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \oplus \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1+4x_2+4x_3 \\ 2y_1+4y_2+4y_3 \\ 2z_1+4z_2+4z_3 \end{bmatrix},$$

which do not agree for all vectors.

4. Since

$$c\odot\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = c\odot\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right) = \left(\begin{bmatrix} x_1+y_1+c \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right)$$

but

$$c\odot\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \oplus c\odot\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1+y_1+2c \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix},$$

so $c\odot(\mathbf{u}+\mathbf{v})$ and $c\odot\mathbf{u}\oplus c\odot\mathbf{v}$ do not always agree.

5.

1. $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$ is in $\mathbb{R}^2$.

2. $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} x_2 + x_1 \\ y_2 + y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

3. $\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ y_1 + y_2 + y_3 \end{bmatrix}$ and

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + x_3 \\ y_1 + y_2 + y_3 \end{bmatrix}$

4. Let $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$.

5. Let $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $-\mathbf{u} = \begin{bmatrix} -x \\ -y \end{bmatrix}$. Then $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$.

6. $c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ is a vector in $\mathbb{R}^2$.

7. $c\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = c\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} c(x_1 + x_2) \\ c(y_1 + y_2) \end{bmatrix} = \begin{bmatrix} cx_1 + cx_2 \\ cy_1 + cy_2 \end{bmatrix} = c\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$.

8. $(c+d)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (c+d)x \\ (c+d)y \end{bmatrix} = \begin{bmatrix} cx + dx \\ cy + dy \end{bmatrix} = c\begin{bmatrix} x \\ y \end{bmatrix} + d\begin{bmatrix} x \\ y \end{bmatrix}$.

9. $c\left(d\begin{bmatrix} x \\ y \end{bmatrix}\right) = c\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} (cd)x \\ (cd)y \end{bmatrix} = (cd)\begin{bmatrix} x \\ y \end{bmatrix}$.

10. $1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

6.

1. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$ is in $M_{2\times 2}$.

2. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} = \begin{bmatrix} e+a & f+b \\ g+c & h+d \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

3. Since the associative property of addition holds for real numbers for any three 2×2 matrices $M_1 + (M_2 + M_3) = (M_1 + M_2) + M_3$.

4. Let $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$.

5. Let $\mathbf{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $-\mathbf{u} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$. Then $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$.

6. $k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$ is a matrix in $M_{2\times 2}$.

7. $k\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = k\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} = \begin{bmatrix} ka+ke & kb+kf \\ kc+kg & kd+kh \end{bmatrix} = kc\begin{bmatrix} a & b \\ c & d \end{bmatrix} + k\begin{bmatrix} e & f \\ g & h \end{bmatrix}$.

8. Since real numbers have the distributive property, $(k+l)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = k\begin{bmatrix} a & b \\ c & d \end{bmatrix} + l\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

9. Since the real numbers have the associative property, $k\left(l\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (kl)\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

10. $1\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

7. Since $(c+d) \odot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+c+d \\ y \end{bmatrix}$ does not equal

$$c \odot \begin{bmatrix} x \\ y \end{bmatrix} + d \odot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+c \\ y \end{bmatrix} + \begin{bmatrix} x+d \\ y \end{bmatrix} = \begin{bmatrix} 2x+c+d \\ 2y \end{bmatrix},$$

for all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$, then V is not a vector space.

8. a. Since $\begin{bmatrix} a \\ b \\ 1 \end{bmatrix} + \begin{bmatrix} c \\ d \\ 1 \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \\ 2 \end{bmatrix}$, the sum of two vectors in V is not another vector in V, then V is not a vector space. **b.** If the third component always remains 1, then to show V is a vector space is equivalent to showing $\mathbb{R}^2$ is a vector space with the standard componentwise operations.

9. Since the operation $\oplus$ is not commutative, then V is not a vector space.

10. The set V with the standard operations is a vector space.

11. The zero vector is given by $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since this vector is not in V, then V is not a vector space.

12. The set V with the standard operations is a vector space.

13. a. Since V is not closed under vector addition, then V is not a vector space. That is, if two matrices from V are added, then the row two, column two entry of the sum has the value 2 and hence, the sum is not in V. **b.** Each of the ten vector space axioms are satisfied with vector addition and scalar multiplication defined in this way.

14. Suppose A and B are skew symmetric. Since $(A+B)^t = A^t + B^t = -A - B = -(A+B)$ and $(cA)^t = cA^t = -(cA)$, so the set of skew symmetric matrices is closed under addition and scalar multiplication. The other vector space properties also hold and V is a vector space.

15. The set of upper triangular matrices with the standard componentwise operations is a vector space.

16. Suppose A and B are symmetric. Since $(A+B)^t = A^t + B^t = A + B$ and $(cA)^t = cA^t = cA$, so the set of symmetric matrices is closed under addition and scalar multiplication. The other vector space properties also hold and V is a vector space.

17. The set of invertible matrices is not a vector space. Let $A = I$ and $B = -I$. Then $A + B$ is not invertible, and hence not in V.

18. Suppose A and B are idempotent matrices. Since $(AB)^2 = ABAB = A^2B^2 = AB$ if and only if A and B commute, the set of idempotent matrices is not a vector space.

19. If A and C are in V and k is a scalar, then $(A+C)B = AB + BC = \mathbf{0}$, and $(kA)B = k(AB) = \mathbf{0}$, so V is closed under addition and scalar multiplication. All the other required properties also hold since V is a subset of the vector space of all matrices with the same operations. Hence, V is a vector space.

20. The set V is closed under addition and scalar multiplication. Since V is a subset of the vector space of all 2×2 matrices with the standard operations, the other vector space properties also hold for V. So V is a vector space.

21. a. The additive identity is $\mathbf{0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since $A \oplus A^{-1} = AA^{-1} = I$, then the additive inverse of A is A^{-1}. **b.** If $c = 0$, then cA is not in V. Notice also that addition is not commutative, since AB is not always equal to BA.

22. a. Since
$$\begin{bmatrix} t \\ 1+t \end{bmatrix} = \begin{bmatrix} t \\ 1+t \end{bmatrix} + \begin{bmatrix} s \\ 1+s \end{bmatrix} = \begin{bmatrix} t+s \\ 1+(t+s) \end{bmatrix} \Leftrightarrow s = 0,$$
the additive identity is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. **b.** Since the other nine vector space properties also hold, V is a vector space.
c. Since $0 \odot \begin{bmatrix} t \\ 1+t \end{bmatrix} = \begin{bmatrix} 0t \\ 1+0t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the additive identity, then $0 \odot \mathbf{v} = \mathbf{0}$.

23. a. The additive identity is $\mathbf{0} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Let $\mathbf{u} = \begin{bmatrix} 1+a \\ 2-a \\ 3+2a \end{bmatrix}$. Then the additive inverse is $-\mathbf{u} = \begin{bmatrix} 1-a \\ 2+a \\ 3-2a \end{bmatrix}$. **b.** Each of the ten vector space axioms is satisfied. **c.** $0 \odot \begin{bmatrix} 1+t \\ 2-t \\ 3+2t \end{bmatrix} = \begin{bmatrix} 1+0t \\ 2-0t \\ 3+2(0)t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

24. Since S is a subset of $\mathbb{R}^3$ with the same standard operations only vector space axioms (1) and (6) need to be verified since the others are inherited from the vector space $\mathbb{R}^3$. If $\mathbf{w_1}$ and $\mathbf{w_2}$ are in S, then let $\mathbf{w_1} = a\mathbf{u}+b\mathbf{v}$ and $\mathbf{w_2} = c\mathbf{u}+d\mathbf{v}$. Then $\mathbf{w_1} + \mathbf{w_2} = (a+c)\mathbf{u} + (b+d)\mathbf{v}$ and $k(a\mathbf{u} + b\mathbf{v}) = (ka)\mathbf{u} + (kb)\mathbf{v}$ are also in S.

25. Each of the ten vector space axioms is satisfied.

26. The set S is a plane through the origin in $\mathbb{R}^3$, so the sum of vectors in S remains in S and a scalar times a vector in S remains in S. Since the other vector space properties are inherited form the vector space $\mathbb{R}^3$, then S is a vector space.

27. Each of the ten vector space axioms is satisfied.

28. a. Since $\cos(0) = 1$ and $\sin(0) = 0$, the additive identity is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The additive inverse of $\begin{bmatrix} \cos t_1 \\ \sin t_1 \end{bmatrix}$ is $\begin{bmatrix} \cos(-t_1) \\ \sin(-t_1) \end{bmatrix} = \begin{bmatrix} \cos t_1 \\ -\sin t_1 \end{bmatrix}$. **b.** The ten required properties hold making V a vector space. **c.** The additive identity in this case is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since $\cos t$ and $\sin t$ are not both 0 for any value of t, then $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in V, so V is not a vector space.

29. Since $(f+g)(0) = f(0) + g(0) = 1 + 1 = 2$, then V is not closed under addition and hence is not a vector space.

30. Since $c \odot (d \odot f)(x) = c \odot (f(x+d)) = f(x+c+d)$ and $(cd) \odot f(x) = f(x+cd)$, do not agree for all scalars, V is not a vector space.

31. a. The zero vector is given by $f(x+0) = x^3$ and $-f(x+t) = f(x-t)$. **b.** Each of the ten vector space axioms is satisfied.

Exercise Set 3.2

A subset W of a vector space V is a subspace of the vector space if vectors in W, using the same addition and scalar multiplication of V, satisfy the ten vector space properties. That is, W is a vector space. Many of the vector space properties are inherited by W from V. For example, if $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then they are also vectors in V, so that $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$. On the other hand, the additive identity may not be a vector in W, which is a requirement for being a vector space. To show that a subset is a subspace it is sufficient to verify that

if $\mathbf{u}$ and $\mathbf{v}$ are in W and c is a scalar, then $\mathbf{u} + c\mathbf{v}$ is another vector in W.