36. Let $\mathbf{v} \in S_1$. Then $\mathbf{v} = c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k} = c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k} + 0(\mathbf{v_1} + \mathbf{v_2})$, so $\mathbf{v} \in S_2$ and hence, $S_1 \subseteq S_2$. Now let $\mathbf{v} \in S_2$, so

$$\mathbf{v} = c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k} + c_{k+1}(\mathbf{v_1} + \mathbf{v_2}) = (c_1 + c_{k+1})\mathbf{v_1} + (c_2 + c_{k+1})\mathbf{v_2} + \cdots + c_k\mathbf{v_k}$$

and hence, $\mathbf{v} \in S_1$. Therefore, $S_2 \subseteq S_1$. Since both containments hold, $S_1 = S_2$.

37. If $\mathbf{A}_3 = c\mathbf{A}_1$, then $\det(A) = 0$. Since the linear system is assumed to be consistent, then it must have infinitely many solutions.

38. If $\mathbf{A}_3 = \mathbf{A}_1 + \mathbf{A}_2$, then $\det(A) = 0$. Since the linear system is assumed to be consistent, then it must have infinitely many solutions.

39. If $f(x) = e^x$ and $g(x) = e^{\frac{1}{2}x}$, then $f'(x) = e^x = f''(x)$, $g'(x) = \frac{1}{2}e^{\frac{1}{2}x}$, and $g''(x) = \frac{1}{4}e^{\frac{1}{2}x}$. Then $2f'' - 3f' + f = 2e^x - 3e^x + e^x = 0$ and $2g'' - 3g' + g = \frac{1}{2}e^{\frac{1}{2}x} - \frac{3}{2}e^{\frac{1}{2}x} + e^{\frac{1}{2}x} = 0$, and hence $f(x)$ and $g(x)$ are solutions to the differential equation. In a similar manner, for arbitrary constants c_1 and c_2, the function $c_1f(x) + c_2g(x)$ is also a solution to the differential equation.

Exercise Set 2.3

In Section 2.3, the fundamental concept of linear independence is introduced. In $\mathbb{R}^2$ and $\mathbb{R}^3$, two nonzero vectors are linearly independent if and only if they are not scalar multiples of each other, so they do not lie on the same line. To determine whether or not a set of vectors $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ is linearly independent set up the vector equation

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \mathbf{0}.$$

If the only solution to the resulting system of equations is $c_1 = c_2 = \cdots = c_k = 0$, then the vectors are linearly independent. If there is one or more nontrivial solutions, then the vectors are linearly dependent. For example, the coordinate vectors in Euclidean space are linearly independent. An alternative method, for determining linear independence is to form a matrix A with column vectors the vectors to test. The matrix must be a square matrix, so for example, if the vectors are in $\mathbb{R}^4$, then there must be four vectors.

- If $\det(A) \neq 0$, then the vectors are linearly independent.
- If $\det(A) = 0$, then the vectors are linearly dependent.

For example, to determine whether or not the vectors $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$ are linearly independent start with the equation

$$c_1\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This yields the linear system

$$\begin{cases} -c_1 + 2c_2 + 3c_3 &= 0 \\ c_1 + c_2 + 5c_3 &= 0 \\ 3c_1 + 2c_2 - c_3 &= 0 \end{cases} \text{ with augmented matrix } \left[\begin{array}{ccc|c} -1 & 2 & 3 & 0 \\ 1 & 1 & 5 & 0 \\ 3 & 2 & -1 & 0 \end{array}\right] \xrightarrow{\text{which reduces to}} \left[\begin{array}{ccc|c} -1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right].$$

So the only solution is $c_1 = c_2 = c_3 = 0$, and the vectors are linearly independent. Now notice that the coefficient matrix

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 1 & 3 \\ 3 & 2 & -1 \end{bmatrix} \xrightarrow{\text{reduces further to}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that A is row equivalent to the identity matrix. This implies the inverse A^{-1} exists and that $\det(A) \neq 0$. So we could have computed $\det(A)$ to conclude the vectors are linearly independent. In addition, since A^{-1}

exists the linear system $A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$ in $\mathbb{R}^3$. So ever vector in $\mathbb{R}^3$ can be written uniquely as a linear combination of the three vectors. The uniqueness is a key result of the linear independence. Other results that aid in making a determination of linear independence or dependence of a set S are:

- If the zero vector is in S, then S is linearly dependent.
- If S consists of m vectors in $\mathbb{R}^n$ and $m > n$, then S is linearly dependent.
- At least one vector in S is a linear combination of other vectors in S if and only if S is linearly dependent.
- Any subset of a set of linearly independent vectors is linearly independent.
- If S a linearly dependent set and is contained by another set T, then T is also linearly dependent.

Solutions to Exercises

1. Since $\begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} = 1$, the vectors are linearly independent.

2. Since $\begin{vmatrix} 2 & -4 \\ 1 & 2 \end{vmatrix} = 8$, the vectors are linearly independent.

3. Since $\begin{vmatrix} 1 & -4 \\ -2 & 8 \end{vmatrix} = 0$, the vectors are linearly dependent.

4. Since $\mathbf{v_3} = \mathbf{v_1} + \mathbf{v_2}$, the vectors are linearly independent.

5. To solve the linear system $c_1\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, we have that $\begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} -1 & 2 \\ 0 & 6 \\ 0 & 0 \end{bmatrix}$, so the only solution is the trivial solution and hence, the vectors are linearly independent.

6. Since $\begin{bmatrix} 4 & -2 \\ 2 & -1 \\ -6 & 3 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 4 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, the vectors are linearly dependent. Also, $\mathbf{v_2} = -\frac{1}{2}\mathbf{v_1}$.

7. Since $\begin{vmatrix} -4 & -5 & 3 \\ 4 & 3 & -5 \\ -1 & 3 & 5 \end{vmatrix} = 0$, the vectors are linearly dependent.

8. Since $\begin{vmatrix} 3 & -1 & -1 \\ -3 & 2 & 3 \\ -1 & -2 & 1 \end{vmatrix} = 16$, the vectors are linearly independent.

9. Since $\begin{bmatrix} 3 & 1 & 3 \\ -1 & 0 & -1 \\ -1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 3 & 1 & 3 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, the linear system $c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \mathbf{0}$ has only the trivial solution and hence, the vectors are linearly independent.

10. Since $\begin{bmatrix} -2 & 3 & -1 \\ -4 & -4 & -12 \\ 1 & 0 & 2 \\ 1 & 4 & 6 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} -2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the linear system $c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \mathbf{0}$ has infinitely many solutions and hence, the vectors are linearly dependent.

11. From the linear system $c_1\begin{bmatrix}3 & 3\\2 & 1\end{bmatrix}+c_2\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}+c_3\begin{bmatrix}1 & -1\\-1 & -2\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$, we have that $\begin{bmatrix}3 & 0 & 1\\3 & 1 & -1\\2 & 0 & -1\\1 & 0 & -2\end{bmatrix}\longrightarrow\begin{bmatrix}3 & 0 & 1\\0 & 1 & -2\\0 & 0 & -5/3\\0 & 0 & 0\end{bmatrix}$. Since the homogeneous linear system has only the trivial solution, then the matrices are linearly independent.

12. Since $\begin{bmatrix}-1 & 1 & 2\\2 & 4 & 2\\1 & 0 & -1\\1 & 1 & 0\end{bmatrix}\xrightarrow{\text{reduces to}}\begin{bmatrix}-1 & 1 & 2\\0 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$, the matrices are linearly dependent.

13. Since $\begin{bmatrix}1 & 0 & -1 & 1\\-2 & -1 & 1 & 1\\-2 & 2 & -2 & -1\\-2 & 2 & 2 & -2\end{bmatrix}\xrightarrow{\text{reduces to}}\begin{bmatrix}1 & 0 & -1 & 1\\0 & -1 & -1 & 3\\0 & 0 & -6 & 7\\0 & 0 & 0 & 11/3\end{bmatrix}$, the homogeneous linear system $c_1M_1+c_2M_2+c_3M_3+c_4M_4=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ has only the trivial solution and hence, the matrices are linearly independent.

14. Since $\begin{bmatrix}0 & -2 & 2 & -2\\-1 & -1 & 0 & 2\\-1 & 1 & -1 & 2\\1 & -1 & 2 & -1\end{bmatrix}\xrightarrow{\text{reduces to}}\begin{bmatrix}-1 & -1 & 0 & 2\\0 & -2 & 2 & -2\\0 & 0 & 1 & -2\\0 & 0 & 0 & 1\end{bmatrix}$, the homogeneous linear system $c_1M_1+c_2M_2+c_3M_3+c_4M_4=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ has only the trivial solution and hence, the matrices are linearly independent.

15. Since $\mathbf{v}_2=-\frac{1}{2}\mathbf{v}_1$, the vectors are linearly dependent.

16. Any set of three or more vectors in $\mathbb{R}^2$ is linearly dependent.

17. Any set of vectors containing the zero vector is linearly dependent.

18. Since $\mathbf{v}_3=\mathbf{v}_1+\mathbf{v}_2$, the vectors are linearly dependent.

19. a. Since $\mathbf{A}_2=-2\mathbf{A}_1$, the column vectors of A are linearly dependent. **b.** Since $\mathbf{A}_3=\mathbf{A}_1+\mathbf{A}_2$, the column vectors of A are linearly dependent.

20. a. Any set of four or more vectors in $\mathbb{R}^3$ is linearly dependent. **b.** Since $\mathbf{A}_3=-\mathbf{A}_1+\mathbf{A}_2$, the column vectors of A are linearly dependent.

21. Form the matrix with column vectors the three given vectors, that is, let $A=\begin{bmatrix}1 & -1 & 2\\2 & 0 & a\\1 & 1 & 4\end{bmatrix}$. Since $\det(A)=-2a+12$, then the vectors are linearly independent if and only if $-2a+12\neq 0$, that is $a\neq 6$.

22. Since the matrix $\begin{bmatrix}1 & 1 & 1\\2 & 0 & -4\\0 & 1 & a\\1 & 0 & -2\end{bmatrix}\xrightarrow{\text{reduces to}}\begin{bmatrix}1 & 1 & 1\\0 & -2 & -6\\0 & 0 & a-3\\0 & 0 & 0\end{bmatrix}$, if $a\neq 3$, then the matrices are linearly independent.

23. a. Since $\begin{vmatrix}1 & 1 & 1\\1 & 2 & 1\\1 & 3 & 2\end{vmatrix}=1$, the vectors are linearly independent. **b.** Since $\left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\1 & 2 & 1 & 1\\1 & 3 & 2 & 3\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & -1\\0 & 0 & 1 & 3\end{array}\right]$ the corresponding linear system has the unique solution $(0,-1,3)$. Hence $\mathbf{v}=-\mathbf{v}_2+3\mathbf{v}_3$.

24. a. Since $\left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, the matrices are linearly independent.
b. Since $\left[\begin{array}{rrr|r} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 5 \\ -1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$, the matrix $M = M_1 + 2M_2 + 3M_3$. **c.** Since $\left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ -1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ the linear system is inconsistent and hence, M can not be written as a linear combination of M_1, M_2 and M_3.

25. Since $\begin{vmatrix} 1 & 2 & 0 \\ -1 & 0 & 3 \\ 2 & 1 & 2 \end{vmatrix} = 13$, the matrix A is invertible, so that $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$.

26. Since $\begin{vmatrix} 3 & 2 & 4 \\ 1 & -1 & 4 \\ 0 & 2 & -4 \end{vmatrix} = 4$, the matrix A is invertible, so that $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$.

27. Since the equation $c_1(1) + c_2(-2 + 4x^2) + c_3(2x) + c_4(-12x + 8x^3) = 0$, for all x, gives that $c_1 = c_2 = c_3 = c_4 = 0$, the polynomials are linear independent.

28. Since the equation $c_1(1) + c_2(x) + c_3(5 + 2x - x^2) = 0$, for all x, gives that $c_1 = c_2 = c_3 = 0$, the polynomials are linear independent.

29. Since the equation $c_1(2) + c_2(x) + c_3(x^2) + c_4(3x - 1) = 0$, for all x, gives that $c_1 = \frac{1}{2}c_4, c_2 = -3c_4, c_3 = 0, c_4 \in \mathbb{R}$, the polynomials are linearly dependent.

30. Since the equation $c_1(x^3 - 2x^2 + 1) + c_2(5x) + c_3(x^2 - 4) + c_4(x^3 + 2x) = 0$, for all x, gives that $c_1 = c_2 = c_3 = c_4 = 0$, the polynomials are linear independent.

31. In the equation $c_1 \cos \pi x + c_2 \sin \pi x = 0$, if $x = 0$, then $c_1 = 0$, and if $x = \frac{1}{2}$, then $c_2 = 0$. Hence, the functions are linearly independent.

32. Consider the equation $c_1 e^x + c_2 e^{-x} + c_3 e^{2x} = 0$ for all x. Let $x = 0, x = \ln 2$, and $x = \ln \frac{5}{2}$ to obtain the linear system $\begin{cases} c_1 + c_2 + c_3 & = 0 \\ 2c_1 + \frac{1}{2}c_2 + 4c_3 & = 0 \\ \frac{5}{2}c_1 + \frac{2}{5}c_2 + \frac{25}{4}c_3 & = 0. \end{cases}$ Since the only solution is the trivial solution the functions are linearly independent.

33. In the equation $c_1 x + c_2 x^2 + c_3 e^x = 0$, if $x = 0$, then $c_3 = 0$. Now let $x = 1$, and $x = -1$, which gives the linear system $\begin{cases} c_1 + c_2 & = 0 \\ -c_1 + c_2 & = 0 \end{cases}$. This system has solution $c_1 = 0$ and $c_2 = 0$. Hence the functions are linearly independent.

34. Consider the equation $c_1 x + c_2 e^x + c_3 \sin \pi x = 0$ for all x. Let $x = 1, x = 0$, and $x = \frac{1}{2}$ to obtain the linear system $\begin{cases} c_1 + ec_2 & = 0 \\ c_2 & = 0 \\ \frac{1}{2}c_1 + e^{1/2}c_2 + c_3 & = 0. \end{cases}$ Since the only solution is the trivial solution the functions are linearly independent.

35. Suppose $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. Then there are scalars c_1 and c_2, not both zero, such that $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$. If $a \neq 0$, then $\mathbf{u} = -\frac{b}{a}\mathbf{v}$. Conversely, suppose there is a scalar c such that $\mathbf{u} = c\mathbf{v}$. Then $\mathbf{u} - c\mathbf{v} = \mathbf{0}$ and hence, the vectors are linearly dependent.

36. Consider the equation $c_1\mathbf{w_1} + c_2\mathbf{w_2} + c_3\mathbf{w_3} = \mathbf{0}$. Since $\mathbf{w_1} = \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}, \mathbf{w_2} = \mathbf{v_2} + \mathbf{v_3}$, and $\mathbf{w_3} = \mathbf{v_3}$, then

$$c_1(\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}) + c_2(\mathbf{v_2} + \mathbf{v_3}) + c_3\mathbf{w_3} = \mathbf{0} \Leftrightarrow c_1\mathbf{v_1} + (c_1 + c_2)\mathbf{v_2} + (c_1 + c_2 + c_3)\mathbf{v_3} = \mathbf{0}.$$

Since S is linearly independent, then $c_1 = 0, c_1 + c_2 = 0, c_1 + c_2 + c_3 = 0$ and hence, the only solution is the trivial solution. Therefore, the set T is linearly independent.

37. Setting a linear combination of $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ to $\mathbf{0}$, we have

$$\mathbf{0} = c_1\mathbf{w_1} + c_2\mathbf{w_2} + c_3\mathbf{w_3} = c_1\mathbf{v_1} + (c_1 + c_2 + c_3)\mathbf{v_2} + (-c_2 + c_3)\mathbf{v_3}.$$

Since the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linear independent, then $c_1 = 0, c_1 + c_2 + c_3 = 0$, and $-c_2 + c_3 = 0$. The only solution to this linear system is the trivial solution $c_1 = c_2 = c_3 = 0$, and hence, the vectors $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ are linearly independent.

38. Consider the equation $c_1\mathbf{w_1} + c_2\mathbf{w_2} + c_3\mathbf{w_3} = \mathbf{0}$. Since $\mathbf{w_1} = \mathbf{v_2}, \mathbf{w_2} = \mathbf{v_1} + \mathbf{v_3}$, and $\mathbf{w_3} = \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}$, then

$$c_1(\mathbf{v_2}) + c_2(\mathbf{v_1} + \mathbf{v_3}) + c_3(\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}) = \mathbf{0} \Leftrightarrow (c_2 + c_3)\mathbf{v_1} + (c_1 + c_3)\mathbf{v_2} + (c_2 + c_3)\mathbf{v_3} = \mathbf{0}.$$

Since S is linearly independent, then $c_2 + c_3 = 0, c_1 + c_3 = 0, c_2 + c_3 = 0$, which implies $c_1 = c_2 = -c_3$. Therefore, the set T is linearly dependent.

39. Consider $c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \mathbf{0}$, which is true if and only if $c_3\mathbf{v_3} = -c_1\mathbf{v_1} - c_2\mathbf{v_2}$. If $c_3 \neq 0$, then $\mathbf{v_3}$ would be a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$ contradicting the hypothesis that it is not the case. Therefore, $c_3 = 0$. Now since $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent $c_1 = c_2 = 0$.

40. a. $\mathbf{v_1} = \mathbf{v_3} - \mathbf{v_2}$, $\mathbf{v_1} = 2\mathbf{v_3} - 2\mathbf{v_2} - \mathbf{v_1}$, $\mathbf{v_1} = 3\mathbf{v_3} - 3\mathbf{v_2} - 2\mathbf{v_1}$ **b.** Consider the equation

$$\mathbf{v_1} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} \Leftrightarrow (1 - c_1)\mathbf{v_1} - c_2\mathbf{v_2} - c_3(\mathbf{v_1} + \mathbf{v_2}) = \mathbf{0} \Leftrightarrow (1 - c_1 - c_3)\mathbf{v_1} + (-c_2 - c_3)\mathbf{v_2} = \mathbf{0}.$$

Then all solutions are given by $c_1 = 1 - c_3, c_2 = -c_3, c_3 \in \mathbb{R}$.

41. Since $\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n}$ are linearly independent, if

$$A\mathbf{x} = x_1\mathbf{A_1} + \cdots + x_n\mathbf{A_n} = \mathbf{0},$$

then $x_1 = x_2 = \cdots = x_n = 0$. Hence, the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

42. Consider

$$\mathbf{0} = c_1A\mathbf{v_1} + c_2A\mathbf{v_2} + \cdots + c_kA\mathbf{v_k} = A(c_1\mathbf{v_1}) + A(c_2\mathbf{v_2}) + \cdots + A(c_k\mathbf{v_k}) = A(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k}).$$

Since A is invertible, then the only solution to the last equation is the trivial solution, so $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \mathbf{0}$. Since the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ are linearly independent, then $c_1 = c_2 = \cdots = c_k = 0$ and hence $A\mathbf{v_1}, A\mathbf{v_2}, \ldots, A\mathbf{v_k}$ are linearly independent.

To show that A invertible is necessary, let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Since $\det(A) = 0$, then A is not invertible. Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which are linearly independent. Then $A\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which are linearly dependent.

Review Exercises Chapter 2

1. Since $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$, the column vectors are linearly independent. If $ad - bc = 0$, then the column vectors are linearly dependent.