27. Using the LU factorization $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$, we have that

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -1 & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

28.

$$A^{-1} = (LU)^{-1} = \left(\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & -1 \\ 1 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

29. Suppose

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This gives the system of equations $ad = 0, ae = 1, bd = 1, be + cf = 0$. The first two equations are satisfied only when $a \neq 0$ and $d = 0$. But this incompatible with the third equation.

30. Since A is row equivalent to B there are elementary matrices such that $B = E_m \ldots E_1 A$ and since B is row equivalent to C there are elementary matrices such that $C = D_n \ldots D_1 B$. Then $C = D_n \ldots D_1 B = D_n \ldots D_1 E_m \ldots E_1 A$ and hence, A is row equivalent to C.

31. If A is invertible, there are elementary matrices $E_1, \ldots, E_k$ such that $I = E_k \cdots E_1 A$. Similarly, there are elementary matrices $D_1, \ldots, D_\ell$ such that $I = D_\ell \cdots D_1 B$. Then $A = E_k^{-1} \cdots E_1^{-1} D_\ell \cdots D_1 B$, so A is row equivalent to B.

32. a. Since L is invertible, the diagonal entries are all nonzero. **b.** The determinant of A is the product of the diagonal entries of L and U, that is $\det(A) = \ell_{11} \cdots \ell_{nn} u_{11} \cdots u_{nn}$. **c.** Since L is lower triangular and invertible it is row equivalent to the identity matrix and can be reduced to I using only replacement operations.

Exercise Set 1.8

1. We need to find positive whole numbers x_1, x_2, x_3, and x_4 such that $x_1 Al_3 + x_2 CuO \longrightarrow x_3 Al_2O_3 + x_4 Cu$ is balanced. That is, we need to solve the linear system

$$\begin{cases} 3x_1 &= 2x_3 \\ x_2 &= 3x_3 \\ x_2 &= x_4 \end{cases}$$, which has infinitely many solutions given by $x_1 = \frac{2}{9}x_2, x_3 = \frac{1}{3}x_2, x_4 = x_2, x_2 \in \mathbb{R}$.

A particular solution that balances the equation is given by $x_1 = 2, x_2 = 9, x_3 = 3, x_4 = 9$.

2. To balance the equation $x_1 I_2 + x_2 Na_2S_2O_3 \longrightarrow x_3 NaI + x_4 Na_2S_4O_6$, we solve the linear system $$\begin{cases} 2x_1 &= x_3 \\ 2x_2 &= x_3 + 2x_4 \\ 2x_2 &= 4x_4 \\ 3x_2 &= 6x_4 \end{cases}$$, so that $x_1 = x_4, x_2 = 2x_4, x_3 = 2x_4, x_4 \in \mathbb{R}$. For a particular solution that balances the equation, let $x_4 = 1$, so $x_1 = 1, x_2 = 2$, and $x_3 = 2$.

3. We need to find positive whole numbers x_1, x_2, x_3, x_4 and x_5 such that

$$x_1\text{NaHCO}_3 + x_2\text{C}_6\text{H}_8\text{O}_7 \longrightarrow x_3\text{Na}_3\text{C}_6\text{H}_5\text{O}_7 + x_4\text{H}_2\text{O} + x_5\text{CO}_2.$$

The augmented matrix for the resulting homogeneous linear system and the reduced row echelon form are

$$\left[\begin{array}{ccccc|c} 1 & 0 & -3 & 0 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 & 0 \\ 1 & 6 & -6 & 0 & -1 & 0 \\ 3 & 7 & -7 & -1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right].$$

Hence the solution set for the linear system is given by $x_1 = x_5, x_2 = \frac{1}{3}x_5, x_3 = \frac{1}{3}x_5, x_4 = x_5, x_5 \in \mathbb{R}$. A particular solution that balances the equation is $x_1 = x_4 = x_5 = 3, x_2 = x_3 = 1$.

4. To balance the equation

$$x_1\text{MnS} + x_2\text{As}_2\text{Cr}_{10}\text{O}_{35} + x_3\text{H}_2\text{SO}_4 \longrightarrow x_4\text{HMnO}_4 + x_5\text{AsH}_3 + x_6\text{CrS}_3\text{O}_{12} + x_7\text{H}_2\text{O},$$

we solve the linear system

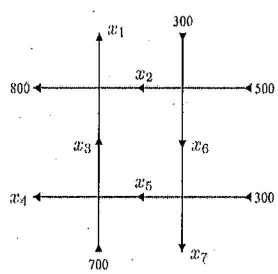
$$\begin{cases} x_1 & = x_4 \\ x_1 + x_3 & = 3x_6 \\ 2x_2 & = x_5 \\ 10x_2 & = x_6 \\ 35x_2 + 4x_3 & = 4x_4 + 12x_6 + x_7 \\ 2x_3 & = x_4 + 3x_5 + 2x_7 \end{cases}$$

The augmented matrix for the equivalent homogeneous linear system

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -3 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 35 & 4 & -4 & 0 & -12 & -1 & 0 \\ 0 & 0 & 2 & -1 & -3 & 0 & -2 & 0 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & -1 & 0 & 0 & -\frac{16}{327} & 0 \\ 0 & 1 & & 0 & 0 & 0 & -\frac{13}{327} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{374}{327} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{16}{327} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{26}{327} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{130}{327} & 0 \end{array}\right].$$

A particular solution that balances the equation is $x_7 = 327, x_1 = 16, x_2 = 13, x_3 = 374, x_4 = 16, x_5 = 26$, and $x_6 = 130$.

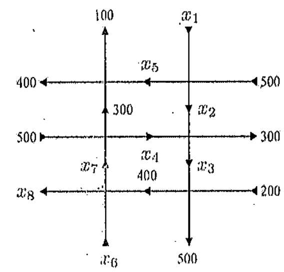
5. Let $x_1, x_2, \ldots, x_7$ be defined as in the figure. The total flows in and out of the entire network and in and out each intersection are given in the table.



Flow In	Flow Out
700+300+500+300	$x_1 + 800 + x_4 + x_7$
$x_2 + x_3$	$x_1 + 800$
$x_5 + 700$	$x_3 + x_4$
$x_6 + 300$	$x_5 + x_7$
500+300	$x_2 + x_6$

Equating the total flows in and out gives a linear system with solution $x_1 = 1000 - x_4 - x_7, x_2 = 800 - x_6, x_3 = 1000 - x_4 + x_6 - x_7, x_5 = 300 + x_6 - x_7$, with x_4, x_6, and x_7 free variables. Since the network consists of one way streets, the individual flows are nonnegative. As a sample solution let, $x_4 = 200, x_6 = 300, x_7 = 100$, then $x_1 = 700, x_2 = 500, x_3 = 1000, x_5 = 500$.

6. Let $x_1, x_2, \ldots, x_8$ be defined as in the figure.



Balancing all in and out flows generates the linear system

$$\begin{cases} x_1 + 500 + x_6 + 200 + 500 & = 100 + 400 + x_8 + 500 + 300 \\ x_1 + 500 & = x_2 + x_5 \\ x_5 + 300 & = 100 + 400 \\ x_7 + 500 & = x_4 + 300 \\ x_6 + 400 & = x_7 + x_8 \\ x_3 + 200 & = 400 + 500 \\ x_2 + x_4 & = x_3 + 300. \end{cases}$$

The set of solutions is given by $x_1 = 500 - x_7, x_2 = 800 - x_7, x_3 = 700, x_4 = 200 + x_7, x_5 = 200, x_6 = -400 + x_7 + x_8, x_7, x_8 \in \mathbb{R}$. In order to have all positive flows, for example, let $x_7 = 300$ and $x_8 = 200$, so $x_1 = 200, x_2 = 500, x_3 = 700, x_4 = 500, x_5 = 200, x_6 = 100$.

7. Equating total flows in and out gives the linear system

$$\begin{cases} x_1   + x_4 & = 150 \\ x_1 - x_2 - x_5 & = 100 \\ x_2 + x_3 & = 100 \\ - x_3 + x_4 + x_5 & = -50 \end{cases}$$

with solution $x_1 = 150 - x_4, x_2 = 50 - x_4 - x_5$, and $x_3 = 50 + x_4 + x_5$. Letting, for example, $x_4 = x_5 = 20$ gives the particular solution $x_1 = 130, x_2 = 10, x_3 = 90$.

8. The set of solutions is given by $x_1 = 200 + x_8, x_2 = -100 + x_8, x_3 = 100 + x_8, x_4 = x_8, x_5 = 150 + x_8, x_6 = -150 + x_8, x_7 = 100 + x_8, x_8 \in \mathbb{R}$. If $x_8 \geq 150$, then all the flows will remain positive.

9. If x_1, x_2, x_3, and x_4 denote the number of grams required from each of the four food groups, then the specifications yield the linear system

$$\begin{cases} 20x_1 + 30x_2 + 40x_3 + 10x_4 & = 250 \\ 40x_1 + 20x_2 + 35x_3 + 20x_4 & = 300 \\ 50x_1 + 40x_2 + 10x_3 + 30x_4 & = 400 \\ 5x_1 + 5x_2 + 10x_3 + 5x_4 & = 70 \end{cases}$$

The solution is $x_1 = 1.4, x_2 = 3.2, x_3 = 1.6, x_4 = 6.2$.

10. If x_1, x_2, and x_3 denote the number of grams required from each of the three food groups, then the specifications yield the linear system

$$\begin{cases} 200x_1 + 400x_2 + 300x_3 & = 2400 \\ 300x_1 + 500x_2 + 400x_3 & = 3500 \\ 40x_1 + 50x_2 + 20x_3 & = 200 \\ 5x_1 + 3x_2 + 2x_3 & = 25 \end{cases}$$

which is inconsistent, and hence it is not possible to prepare the required diet.