48.

$$x = \frac{\begin{vmatrix} -12 & -7 \\ 5 & 11 \end{vmatrix}}{\begin{vmatrix} -10 & -7 \\ 12 & 11 \end{vmatrix}} = \frac{97}{26},$$

$$y = \frac{\begin{vmatrix} -10 & -12 \\ 12 & 5 \end{vmatrix}}{\begin{vmatrix} -10 & -7 \\ 12 & 11 \end{vmatrix}} = -\frac{47}{23}$$

49.

$$x = \frac{\begin{vmatrix} 4 & -3 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} -1 & -3 \\ -8 & 4 \end{vmatrix}} = -\frac{25}{28},$$

$$y = \frac{\begin{vmatrix} -1 & 4 \\ -8 & 3 \end{vmatrix}}{\begin{vmatrix} -1 & -3 \\ -8 & 4 \end{vmatrix}} = -\frac{29}{28}$$

50. $x = \frac{\begin{vmatrix} -8 & 1 & -4 \\ 3 & -4 & 1 \\ -8 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} -2 & 1 & -4 \\ 0 & -4 & 1 \\ 4 & 0 & -1 \end{vmatrix}} = -\frac{91}{68}, y = \frac{\begin{vmatrix} -2 & -8 & -4 \\ 0 & 3 & 1 \\ 4 & -8 & -1 \end{vmatrix}}{\begin{vmatrix} -2 & 1 & -4 \\ 0 & -4 & 1 \\ 4 & 0 & -1 \end{vmatrix}} = -\frac{3}{34}, z = \frac{\begin{vmatrix} -2 & 1 & -8 \\ 0 & -4 & 3 \\ 4 & 0 & -8 \end{vmatrix}}{\begin{vmatrix} -2 & 1 & -4 \\ 0 & -4 & 1 \\ 4 & 0 & -1 \end{vmatrix}} = \frac{45}{17}$

51. $x = \frac{\begin{vmatrix} -2 & 3 & 2 \\ -2 & -3 & -8 \\ 2 & 2 & -7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 2 \\ -1 & -3 & -8 \\ -3 & 2 & -7 \end{vmatrix}} = -\frac{160}{103}, y = \frac{\begin{vmatrix} 2 & -2 & 2 \\ -1 & -2 & -8 \\ -3 & 2 & -7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 2 \\ -1 & -3 & -8 \\ -3 & 2 & -7 \end{vmatrix}} = \frac{10}{103}, z = \frac{\begin{vmatrix} 2 & 3 & -2 \\ -1 & -3 & -2 \\ -3 & 2 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 2 \\ -1 & -3 & -8 \\ -3 & 2 & -7 \end{vmatrix}} = \frac{42}{103}$

52. Suppose $A^t = -A$. Then $\det(A) = \det(A^t) = \det(-A) = (-1)^n \det(A)$. If n is odd, then $\det(A) = -\det(A)$ and hence $\det(A) = 0$. Therefore, A is not invertible.

53. Expansion of the determinant of A across row one equals the expansion down column one of A^t, so $\det(A) = \det(A^t)$.

54. If $A = (a_{ij})$ is upper triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$. But A^t is lower triangular with the same diagonal entries, so $\det(A^t) = a_{11}a_{22}\cdots a_{nn} = \det(A)$.

Exercise Set 1.7

A factorization of a matrix, like factoring a quadratic polynomial, refers to writing a matrix as the product of other matrices. Just like the resulting linear factors of a quadratic are useful and provide information about the original quadratic polynomial, the lower triangular and upper triangular factors in an *LU* factorization are easier to work with and can be used to provide information about the matrix. An elementary matrix is obtained by applying one row operation to the identity matrix. For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-1)R_1 + R_3 \to R_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{\text{Elementary Matrix}}.$$

If a matrix A is multiplied by an elementary matrix E, the result is the same as applying to the matrix A the corresponding row operation that defined E. For example, using the elementary matrix above

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Also since each elementary row operation can be reversed, elementary matrices are invertible. To find an *LU* factorization of A :

- Row reduce the matrix A to an upper triangular matrix U.
- Use the corresponding elementary matrices to write U in the form $U = E_k \cdots E_1 A$.
- If row interchanges are not required, then each of the elementary matrices is lower triangular, so that $A = E_1^{-1} \cdots E_k^{-1} U$ is an LU factorization of A. If row interchanges are required, then a permutation matrix is also required.

When $A = LU$ is an LU factorization of A, and A is invertible, then $A = (LU)^{-1} = U^{-1}L^{-1}$. If the determinant of A is required, then since L and U are triangular, their determinants are simply the product of their diagonal entries and $\det(A) = \det(LU) = \det(L)\det(U)$. An LU factorization can also be used to solve a linear system. To solve the linear system

$$\begin{cases} x_1 - x_2 + 2x_2 & = 2 \\ 2x_1 + 2x_2 + x_3 & = 0 \\ -x_1 + x_2 & = 1 \end{cases}$$

the first step is to find an LU factorization of the coefficient matrix of the linear system. That is,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -3 \\ 0 & 0 & 2 \end{bmatrix}.$$

Next solve the linear system $L\mathbf{y} = \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ using forward substitution, so that $y_1 = 2, y_2 = -2y_1 = -4, y_3 = 1 + y_1 = 3$. As the final step solve $U\mathbf{x} = \mathbf{y} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$ using back substitution, so that $x_3 = \frac{3}{2}, x_2 = \frac{1}{4}(-4 + 3x_3) = \frac{1}{8}, x_1 = 2 + x_2 - 2x_3 = -\frac{7}{8}$.

Solutions to Exercises

1. a. $E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b. $EA = \begin{bmatrix} 1 & 2 & 1 \\ 5 & 5 & 4 \\ 1 & 1 & -4 \end{bmatrix}$

2. a. $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b. $EA = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & -4 \end{bmatrix}$

3. a. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

b. $EA = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ -8 & -2 & -10 \end{bmatrix}$

4. a. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

b. $EA = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & -1 & -5 \end{bmatrix}$

5. a. The required row operations are $2R_1 + R_2 \to R_2$, $\frac{1}{10}R_2 \to R_2$, and $-3R_2 + R_1 \to R_1$. The corresponding elementary matrices that transform A to the identity are given in

$$I = E_3E_2E_1A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} A.$$

b. Since elementary matrices are invertible, we have that

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

6. a. The required row operations are $R_1 + R_2 \to R_2$, $\frac{1}{10}R_2 \to R_2$, $-5R_2 + R_1 \to R_1$, and $-\frac{1}{2}R_1 \to R_1$. The corresponding elementary matrices that transform A to the identity are given in

$$I = E_4E_3E_2E_1A = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} A.$$

b. Since elementary matrices are invertible, we have that

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}.$$

7. a. The identity matrix can be written as $I = E_5E_4E_3E_2E_1A$, where the elementary matrices are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_4 = \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}.$$ **b.** $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}$

8. a. Row operations to reduce the matrix to the identity matrix are

$$\begin{array}{ccc} 3R_1 + R_2 \to R_2 & -2R_1 + R_3 \to R_3 & R_2 \leftrightarrow R_3 \\ 4R_2 + R_3 \to R_3 & -R_1 \to R_1 & -R_2 \to R_2 \\ -R_3 \to R_3 & R_2 + R_1 \to R_1 & -R_3 + R_2 \to R_2 \end{array}$$

with corresponding elementary matrices

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix},$$

$$E_5 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, E_8 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$$E_9 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$ **b.** $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}E_8^{-1}E_9^{-1}$

9. a. The identity matrix can be written as $I = E_6 \cdots E_1A$, where the elementary matrices are

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$ **b.** $A = E_1^{-1}E_2^{-1} \cdots E_6^{-1}$

10. a. There are only two row interchanges needed, $R_1 \leftrightarrow R_4$ and $R_2 \leftrightarrow R_3$. So

$$I = E_2E_1A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A.$$

b. $A = E_1^{-1}E_2^{-1}$.

11. The matrix A can be row reduced to an upper triangular matrix $U = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$, by means of the one operation $3R_1 + R_2 \rightarrow R_2$. The corresponding elementary matrix is $E = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, so that $EA = U$. Then the LU factorization of A is $A = LU = E^{-1}U = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

12. $L = \begin{bmatrix} 1 & 0 \\ 1/6 & 1 \end{bmatrix}, U = \begin{bmatrix} 3 & 9 \\ 0 & -1/2 \end{bmatrix}$

13. The matrix A can be row reduced to an upper triangular matrix $U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, by means of the operations $(-2)R_1 + R_2 \rightarrow R_2$ and $3R_1 + R_3 \rightarrow R_3$. The corresponding elementary matrices are $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, so that $E_2E_1A = U$. Then the LU factorization of A is $A = LU = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

14. $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$,

$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

15. $A = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

16. $A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

17. The first step is to determine an LU factorization for the coefficient matrix of the linear system $A = \begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}$. We have that $A = LU = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}$. Next we solve $L\mathbf{y} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ to obtain $y_1 = -1$ and $y_2 = 3$. The last step is to solve $U\mathbf{x} = \mathbf{y}$, which has the unique solution $x_1 = 2$ and $x_2 = 3$.

18. An LU factorization of the matrix A is given by $A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$. Then the solution to $L\mathbf{y} = \begin{bmatrix} 2 \\ -7/2 \end{bmatrix}$ is $y_1 = 2, y_2 = \frac{1}{2}$, so the solution to $U\mathbf{x} = \mathbf{y}$ is $x_1 = 1, x_2 = \frac{1}{2}$.

19. An LU factorization of the coefficient matrix A is $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. To solve $L\mathbf{y} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$, we have that

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -3 \\ 2 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 2 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

and hence, the solution is $y_1 = 0, y_2 = -3$, and $y_3 = 1$. Finally, the solution to $U\mathbf{x} = \mathbf{y}$, which is the solution to the linear system, is $x_1 = 23, x_2 = -5$, and $x_3 = 1$.

20. An LU factorization of the matrix A is given by $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$. Then the solution to $L\mathbf{y} = \begin{bmatrix} -1 \\ 8 \\ 4 \end{bmatrix}$ is $y_1 = -1, y_2 = 10, y_3 = 2$, so the solution to $U\mathbf{x} = \mathbf{y}$ is $x_1 = 1, x_2 = 2, x_3 = 2$.

21. LU Factorization of the coefficient matrix:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution to $L\mathbf{y} = \begin{bmatrix} 5 \\ 6 \\ 14 \\ -8 \end{bmatrix}$: $y_1 = 5, y_2 = 1, y_3 = 4, y_4 = -2$

Solution to $U\mathbf{x} = \mathbf{y}$: $x_1 = -25, x_2 = -7, x_3 = 6, x_4 = -2$

22. An LU factorization of the matrix A is given by $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Then the solution to $L\mathbf{y} = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 1 \end{bmatrix}$ is $y_1 = 5, y_2 = -2, y_3 = 6, y_4 = -13$, so the solution to $U\mathbf{x} = \mathbf{y}$ is $x_1 = \frac{31}{2}, x_2 = -34, x_3 = \frac{51}{2}, x_4 = -\frac{13}{2}$.

23. In order to row reduce the matrix A to an upper triangular matrix requires the operation of switching rows. This is reflected in the matrix P in the factorization

$$A = PLU = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 0 & 1 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -\frac{4}{5} \\ 0 & 0 & 1 \end{bmatrix}.$$

24. In order to row reduce the matrix A to an upper triangular matrix requires the operation of switching rows. This is reflected in the matrix P in the factorization

$$A = PLU = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

25. Using the LU factorization $A = LU = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$, we have that

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -11 & -4 \\ 3 & 1 \end{bmatrix}.$$

26.

$$A^{-1} = (LU)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & 6 \end{bmatrix} \right)^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1 & -\frac{7}{6} \\ 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{10}{3} & -\frac{7}{6} \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

27. Using the LU factorization $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$, we have that

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -1 & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

28.

$$A^{-1} = (LU)^{-1} = \left(\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & -1 \\ 1 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

29. Suppose

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This gives the system of equations $ad = 0, ae = 1, bd = 1, be + cf = 0$. The first two equations are satisfied only when $a \neq 0$ and $d = 0$. But this incompatible with the third equation.

30. Since A is row equivalent to B there are elementary matrices such that $B = E_m \ldots E_1 A$ and since B is row equivalent to C there are elementary matrices such that $C = D_n \ldots D_1 B$. Then $C = D_n \ldots D_1 B = D_n \ldots D_1 E_m \ldots E_1 A$ and hence, A is row equivalent to C.

31. If A is invertible, there are elementary matrices $E_1, \ldots, E_k$ such that $I = E_k \cdots E_1 A$. Similarly, there are elementary matrices $D_1, \ldots, D_\ell$ such that $I = D_\ell \cdots D_1 B$. Then $A = E_k^{-1} \cdots E_1^{-1} D_\ell \cdots D_1 B$, so A is row equivalent to B.

32. **a.** Since L is invertible, the diagonal entries are all nonzero. **b.** The determinant of A is the product of the diagonal entries of L and U, that is $\det(A) = \ell_{11} \cdots \ell_{nn} u_{11} \cdots u_{nn}$. **c.** Since L is lower triangular and invertible it is row equivalent to the identity matrix and can be reduced to I using only replacement operations.

Exercise Set 1.8

1. We need to find positive whole numbers x_1, x_2, x_3, and x_4 such that $x_1 Al_3 + x_2 CuO \longrightarrow x_3 Al_2O_3 + x_4 Cu$ is balanced. That is, we need to solve the linear system

$$\begin{cases} 3x_1 &= 2x_3 \\ x_2 &= 3x_3 \\ x_2 &= x_4 \end{cases}$$, which has infinitely many solutions given by $x_1 = \frac{2}{9}x_2, x_3 = \frac{1}{3}x_2, x_4 = x_2, x_2 \in \mathbb{R}$.

A particular solution that balances the equation is given by $x_1 = 2, x_2 = 9, x_3 = 3, x_4 = 9$.

2. To balance the equation $x_1 I_2 + x_2 Na_2S_2O_3 \longrightarrow x_3 NaI + x_4 Na_2S_4O_6$, we solve the linear system

$$\begin{cases} 2x_1 &= x_3 \\ 2x_2 &= x_3 + 2x_4 \\ 2x_2 &= 4x_4 \\ 3x_2 &= 6x_4 \end{cases}$$, so that $x_1 = x_4, x_2 = 2x_4, x_3 = 2x_4, x_4 \in \mathbb{R}$. For a particular solution that balances the equation, let $x_4 = 1$, so $x_1 = 1, x_2 = 2$, and $x_3 = 2$.