$$A \text{ is invertible } \Leftrightarrow \det(A) \neq 0 \Leftrightarrow A\mathbf{x} = \mathbf{b} \text{ has a unique solution}$$
$$\Leftrightarrow A\mathbf{x} = \mathbf{0} \text{ has only the trivial solution}$$
$$\Leftrightarrow A \text{ is row equivalent to } I.$$

One useful observation that follows is that if the determinant of the coefficient matrix is 0, then the linear system is inconsistent or has infinitely many solutions.

Solutions to Exercises

1. Since the matrix is triangular, the determinant is the product of the diagonal entries. Hence the determinant is 24.

2. Since the matrix has two identical rows, the determinant is 0.

3. Since the matrix is triangular, the determinant is the product of the diagonal entries. Hence the determinant is -10.

4. Since the second row is twice the first, the determinant 0.

5. Since the determinant is 2, the matrix is invertible.

6. Since the determinant is -17, the matrix is invertible.

7. Since the matrix is triangular the determinant is -6 and hence, the matrix is invertible.

8. Since there are two identical rows the determinant is 0, and hence the matrix is not invertible.

9. a. Expanding along row one

$$\det(A) = 2\begin{vmatrix} -1 & 4 \\ 1 & -2 \end{vmatrix} - (0)\begin{vmatrix} 3 & 4 \\ -4 & -2 \end{vmatrix} + (1)\begin{vmatrix} 3 & -1 \\ -4 & 1 \end{vmatrix} = -5.$$

b. Expanding along row two

$$\det(A) = -3\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 1 \\ -4 & -2 \end{vmatrix} + (4)\begin{vmatrix} 2 & 0 \\ -4 & 1 \end{vmatrix} = -5.$$

c. Expanding along column two

$$\det(A) = -(0)\begin{vmatrix} 3 & 4 \\ -4 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 1 \\ -4 & -2 \end{vmatrix} + (1)\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = -5.$$

d. $\det\left(\begin{bmatrix} -4 & 1 & -2 \\ 3 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix}\right) = 5$ **e.** Let B denote the matrix in part (d) and B' denote the new matrix. Then $\det(B') = -2\det(B) = -10$. Hence, $\det(A) = \frac{1}{2}\det(B')$. **f.** Let B'' denote the new matrix. The row operation does not change the determinant, so $\det(B'') = \det(B') = -10$. **g.** Since $\det(A) \neq 0$, the matrix A does have an inverse.

10. a. Expanding along row four

$$\det(A) = -3\begin{vmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix} - 3\begin{vmatrix} -1 & 1 & 2 \\ 3 & -2 & -1 \\ 0 & 1 & 1 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 & 1 \\ 3 & -2 & 0 \\ 0 & 1 & 0 \end{vmatrix} = -15.$$

b. Expanding along row three the signs alternate in the pattern $+, -, +, -$, and the determinant is again -15. **c.** Expanding along column two the signs alternate in the pattern $-, +, -, +$ and the determinant is again -15. **d** Since row two contains two zeros this is the preferred expansion. **e.** Since the determinant is nonzero, the matrix is invertible.

11. Determinant: 13; Invertible

12. Determinant: -19; Invertible

13. Determinant: -16; Invertible

14. Determinant: 5; Invertible

15. Determinant: 0; Not invertible

16. Determinant: 0; Not invertible

17. Determinant: 30; Invertible

18. Determinant: -76; Invertible

19. Determinant: -90; Invertible

20. Determinant: 8; Invertible

21. Determinant: 0; Not invertible

22. Determinant: 0; Not invertible

23. Determinant: -32; Invertible

24. Determinant: 59; Invertible

25. Determinant: 0; Not invertible

26. Determinant: 5; Invertible

27. Since multiplying a matrix by a scalar multiplies each row by the scalar, we have that $\det(3A) = 3^3\det(A) = 270$.

28. $\det(2A^{-1}) = 2^3\det(A^{-1}) = \frac{8}{\det(A)} = \frac{4}{5}$

29.

$$\det((2A)^{-1}) = \frac{1}{\det(2A)} = \frac{1}{2^3\det(A)} = \frac{1}{80}$$

30. Since the matrix is the transpose of the original matrix but with two columns interchanged, the determinant is -10.

31. Expanding along row 3

$$\begin{vmatrix} x^2 & x & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix} = (-5)\begin{vmatrix} x^2 & x \\ 2 & 1 \end{vmatrix} = -(5)(x^2-2x) = -5x^2+10x.$$

Then the determinant of the matrix is 0 when $-5x^2+10x = -5x(x-2) = 0$, that is $x=0$ or $x=2$.

32. Since

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{\text{reduces to}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

using only the operation of adding a multiple of one row to another, the determinants of the two matrices are equal. Since the reduced matrix is triangular and the product of the diagonal entries is 1, then the determinant of the original matrix is also 1.

33. Since the determinant is $a_1b_2 - b_1a_2 - xb_2 + xa_2 + yb_1 - ya_1$, then the determinant will be zero precisely when $y = \frac{b_2-a_2}{b_1-a_1}x + \frac{b_1a_2-a_1b_2}{b_1-a_1}$. This equation describes a straight line in the plane.

34. **a.** $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$ **b.** $\det(A) = 0, \det(B) = 0, \det(C) = -4$ **c.** Only the matrix C has an inverse. **d.** Since $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 1 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & -5 \end{array}\right]$, the linear system is inconsistent. **e.** Since $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 6 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$, there are infinitely many solutions given by $x = 3 - y, y \in \mathbb{R}$. **f.** Since $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -2 & 1 \end{array}\right] \xrightarrow{\text{reduces to}} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -4 & -5 \end{array}\right]$, the linear system has the unique solution $x = \frac{7}{4}, y = \frac{5}{4}$.

35. **a.** $A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \\ 2 & -2 & -2 \end{bmatrix}$ **b.** $\det(A) = 2$ **c.** Since the determinant of the coefficient matrix is not zero it is invertible and hence, the linear system has a unique solution. **d.** The unique solution is $\mathbf{x} = \begin{bmatrix} 3 \\ 8 \\ -4 \end{bmatrix}$.

36. **a.** $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & 1 \\ 2 & 6 & -4 \end{bmatrix}$ **b.** $\det(A) = 0$ **c.** The system will have infinitely many solutions or is inconsistent. **d.** Since $\left[\begin{array}{ccc|c} 1 & 3 & -2 & -1 \\ 2 & 5 & 1 & 2 \\ 2 & 6 & -4 & -2 \end{array}\right]$ $\xrightarrow{\text{reduces to}}$ $\left[\begin{array}{ccc|c} 1 & 3 & -2 & -1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$, the linear system has infinitely many solutions given by $x = 11 - 13z, y = -4 + 5z, z \in \mathbb{R}$.

37. **a.** $A = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 0 & 2 \\ 1 & -3 & -3 \end{bmatrix}$ **b.** Expanding along column three, then $\det(A) = \begin{vmatrix} -1 & -1 \\ 2 & 2 \end{vmatrix} = 0$. **c.** Since the determinant of the coefficient matrix is 0, A is not invertible. Therefore, the linear system has either no solutions or infinitely many solutions.
d. Since the augmented matrix reduces to

$$\left[\begin{array}{ccc|c} -1 & 0 & -1 & -1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & -3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

the linear system is inconsistent.

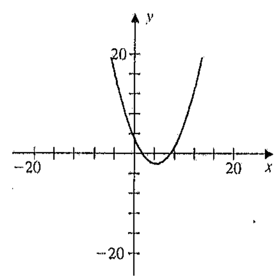
38. **a.** Since

$$\begin{vmatrix} x^2 & x & y & 1 \\ 0 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 \\ 16 & 4 & -2 & 1 \end{vmatrix} = 3x^2 - 27x - 12y + 36,$$

the equation of the parabola is

$$3x^2 - 27x - 12y + 36.$$

b.



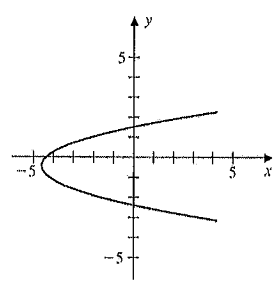
39. **a.** Since

$$\begin{vmatrix} y^2 & x & y & 1 \\ 4 & -2 & -2 & 1 \\ 4 & 3 & 2 & 1 \\ 9 & 4 & -3 & 1 \end{vmatrix} = -29y^2 + 20x - 25y + 106,$$

the equation of the parabola is

$$-29y^2 + 20x - 25y + 106 = 0.$$

b.



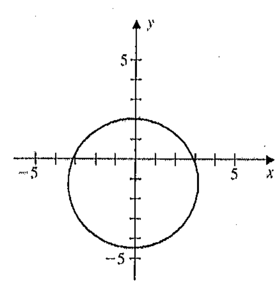
40. **a.** Since

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 18 & -3 & -3 & 1 \\ 5 & -1 & 2 & 1 \\ 9 & 3 & 0 & 1 \end{vmatrix} = -24x^2 - 24y^2 - 6x - 60y + 234,$$

the equation of the circle is

$$-24x^2 - 24y^2 - 6x - 60y + 234 = 0.$$

b.



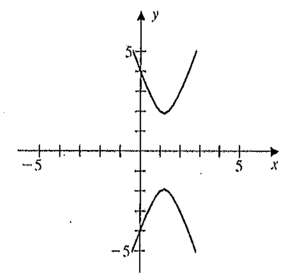
41. **a.** Since

$$\begin{vmatrix} x^2 & y^2 & x & y & 1 \\ 0 & 16 & 0 & -4 & 1 \\ 0 & 16 & 0 & 4 & 1 \\ 1 & 4 & 1 & -2 & 1 \\ 4 & 9 & 2 & 3 & 1 \end{vmatrix} = 136x^2 - 16y^2 - 328x + 256,$$

the equation of the hyperbola is

$$136x^2 - 16y^2 - 328x + 256 = 0.$$

b.



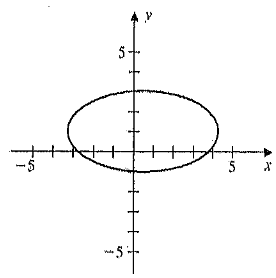
42. **a.** Since

$$\begin{vmatrix} x^2 & y^2 & x & y & 1 \\ 9 & 4 & -3 & 2 & 1 \\ 1 & 9 & -1 & 3 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 16 & 4 & 4 & 2 & 1 \end{vmatrix} = -84x^2 - 294y^2 + 84x + 630y + 924,$$

the equation of the ellipse is

$$-84x^2 - 294y^2 + 84x + 630y + 924 = 0.$$

b.



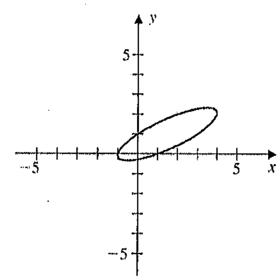
43. **a.** Since

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 4 & 4 & 4 & 2 & 2 & 1 \\ 9 & 3 & 1 & 3 & 1 & 1 \end{vmatrix} = -12 + 12x^2 - 36xy + 42y^2 - 30y,$$

the equation of the ellipse is

$$-12 + 12x^2 - 36xy + 42y^2 - 30y = 0.$$

b.



44.

$$x = \frac{\begin{vmatrix} 4 & 3 \\ 4 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix}} = 2,$$

$$y = \frac{\begin{vmatrix} 2 & 4 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix}} = 0$$

45.

$$x = \frac{\begin{vmatrix} 7 & -5 \\ 6 & -3 \end{vmatrix}}{\begin{vmatrix} 5 & -5 \\ 2 & -3 \end{vmatrix}} = -\frac{9}{5},$$

$$y = \frac{\begin{vmatrix} 5 & 7 \\ 2 & 6 \end{vmatrix}}{\begin{vmatrix} 5 & -5 \\ 2 & -3 \end{vmatrix}} = -\frac{16}{5}$$

46.

$$x = \frac{\begin{vmatrix} 4 & 5 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix}} = \frac{11}{8},$$

$$y = \frac{\begin{vmatrix} 2 & 4 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix}} = \frac{5}{9}$$

47.

$$x = \frac{\begin{vmatrix} 3 & -4 \\ -10 & 5 \end{vmatrix}}{\begin{vmatrix} -9 & -4 \\ -7 & 5 \end{vmatrix}} = \frac{25}{73},$$

$$y = \frac{\begin{vmatrix} -9 & 3 \\ -7 & -10 \end{vmatrix}}{\begin{vmatrix} -9 & -4 \\ -7 & 5 \end{vmatrix}} = -\frac{111}{73}$$

48.

$$x = \frac{\begin{vmatrix} -12 & -7 \\ 5 & 11 \end{vmatrix}}{\begin{vmatrix} -10 & -7 \\ 12 & 11 \end{vmatrix}} = \frac{97}{26},$$

$$y = \frac{\begin{vmatrix} -10 & -12 \\ 12 & 5 \end{vmatrix}}{\begin{vmatrix} -10 & -7 \\ 12 & 11 \end{vmatrix}} = -\frac{47}{23}$$

49.

$$x = \frac{\begin{vmatrix} 4 & -3 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} -1 & -3 \\ -8 & 4 \end{vmatrix}} = -\frac{25}{28},$$

$$y = \frac{\begin{vmatrix} -1 & 4 \\ -8 & 3 \end{vmatrix}}{\begin{vmatrix} -1 & -3 \\ -8 & 4 \end{vmatrix}} = -\frac{29}{28}$$

50. $x = \frac{\begin{vmatrix} -8 & 1 & -4 \\ 3 & -4 & 1 \\ -8 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} -2 & 1 & -4 \\ 0 & -4 & 1 \\ 4 & 0 & -1 \end{vmatrix}} = -\frac{91}{68}, y = \frac{\begin{vmatrix} -2 & -8 & -4 \\ 0 & 3 & 1 \\ 4 & -8 & -1 \end{vmatrix}}{\begin{vmatrix} -2 & 1 & -4 \\ 0 & -4 & 1 \\ 4 & 0 & -1 \end{vmatrix}} = -\frac{3}{34}, z = \frac{\begin{vmatrix} -2 & 1 & -8 \\ 0 & -4 & 3 \\ 4 & 0 & -8 \end{vmatrix}}{\begin{vmatrix} -2 & 1 & -4 \\ 0 & -4 & 1 \\ 4 & 0 & -1 \end{vmatrix}} = \frac{45}{17}$

51. $x = \frac{\begin{vmatrix} -2 & 3 & 2 \\ -2 & -3 & -8 \\ 2 & 2 & -7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 2 \\ -1 & -3 & -8 \\ -3 & 2 & -7 \end{vmatrix}} = -\frac{160}{103}, y = \frac{\begin{vmatrix} 2 & -2 & 2 \\ -1 & -2 & -8 \\ -3 & 2 & -7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 2 \\ -1 & -3 & -8 \\ -3 & 2 & -7 \end{vmatrix}} = \frac{10}{103}, z = \frac{\begin{vmatrix} 2 & 3 & -2 \\ -1 & -3 & -2 \\ -3 & 2 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 2 \\ -1 & -3 & -8 \\ -3 & 2 & -7 \end{vmatrix}} = \frac{42}{103}$

52. Suppose $A^t = -A$. Then $\det(A) = \det(A^t) = \det(-A) = (-1)^n \det(A)$. If n is odd, then $\det(A) = -\det(A)$ and hence $\det(A) = 0$. Therefore, A is not invertible.

53. Expansion of the determinant of A across row one equals the expansion down column one of A^t, so $\det(A) = \det(A^t)$.

54. If $A = (a_{ij})$ is upper triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$. But A^t is lower triangular with the same diagonal entries, so $\det(A^t) = a_{11}a_{22}\cdots a_{nn} = \det(A)$.

Exercise Set 1.7

A factorization of a matrix, like factoring a quadratic polynomial, refers to writing a matrix as the product of other matrices. Just like the resulting linear factors of a quadratic are useful and provide information about the original quadratic polynomial, the lower triangular and upper triangular factors in an LU factorization are easier to work with and can be used to provide information about the matrix. An elementary matrix is obtained by applying one row operation to the identity matrix. For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-1)R_1 + R_3 \to R_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{\text{Elementary Matrix}}.$$

If a matrix A is multiplied by an elementary matrix E, the result is the same as applying to the matrix A the corresponding row operation that defined E. For example, using the elementary matrix above

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Also since each elementary row operation can be reversed, elementary matrices are invertible. To find an LU factorization of A :