33. If $AB = BA$, then $B^{-1}AB = A$, so that $B^{-1}A = AB^{-1}$. Then $(AB^{-1})^t = (B^{-1})^tA^t$. Since $(B^{-1})^tB^t = (BB^{-1})^t = I$, we have that $(B^{-1})^t = (B^t)^{-1}$. Finally, $(AB^{-1})^t = (B^t)^{-1}A^t = B^{-1}A = AB^{-1}$ and hence, AB^{-1} is symmetric.

35. Assuming $A^t = A^{-1}$ and $B^t = B^{-1}$, we need to show that $(AB)^t = (AB)^{-1}$. But $(AB)^t = B^tA^t = B^{-1}A^{-1} = (AB)^{-1}$ and hence, AB is orthogonal.

37. a. Using the associative property of matrix multiplication, we have that

$$(ABC)(C^{-1}B^{-1}A^{-1}) = (AB)CC^{-1}(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AA^{-1} = I.$$

b. The proof is by induction on the number of matrices k.
Base Case: When $k = 2$, since $(A_1A_2)^{-1} = A_2^{-1}A_1^{-1}$, the statement holds.
Inductive Hypothesis: Suppose that $(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_1^{-1}$. Then for $k+1$ matrices, we have that $(A_1A_2\cdots A_kA_{k+1})^{-1} = ([A_1A_2\cdots A_k]A_{k+1})^{-1}$. Since $[A_1A_2\cdots A_k]$ and A_{k+1} can be considered as two matrices, by the base case, we have that $([A_1A_2\cdots A_k]A_{k+1})^{-1} = A_{k+1}^{-1}[A_1A_2\cdots A_k]^{-1}$. Finally, by the inductive hypothesis

$$([A_1A_2\cdots A_k]A_{k+1})^{-1} = A_{k+1}^{-1}[A_1A_2\cdots A_k]^{-1} = A_{k+1}^{-1}A_k^{-1}A_{k-1}^{-1}\cdots A_1^{-1}.$$

39. If A is invertible, then the augmented matrix $[A|I]$ can be row reduced to $[I|A^{-1}]$. If A is upper triangular, then only terms on or above the main diagonal can be affected by the reduction process and hence the inverse is upper triangular. Similarly, the inverse for an invertible lower triangle matrix is also lower triangular.

41. a. Expanding the matrix equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, gives $\begin{bmatrix} ax_1 + bx_3 & ax_2 + bx_4 \\ cx_1 + dx_3 & cx_2 + dx_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. **b.** From part (a), we have the two linear systems

$$\begin{cases} ax_1 + bx_3 & = 1 \\ cx_1 + dx_3 & = 0 \end{cases} \text{ and } \begin{cases} ax_2 + bx_4 & = 0 \\ cx_2 + dx_4 & = 1 \end{cases}.$$

In the first linear system, multiplying the first equation by d and the second by b and then adding the results gives the equation $(ad - bc)x_1 = d$. Since the assumption is that $ad - bc = 0$, then $d = 0$. Similarly, from the second linear system we conclude that $b = 0$. **c.** From part (b), both $b = 0$ and $d = 0$. Notice that if in addition either $a = 0$ or $c = 0$, then the matrix is not invertible. Also from part (b), we have that $ax_1 = 1, ax_2 = 0, cx_1 = 0$, and $cx_2 = 1$. If a and c are not zero, then these equations are inconsistent and the matrix is not invertible.

Exercise Set 1.5

A linear system can be written as a matrix equation $A\mathbf{x} = \mathbf{b}$, where A is the coefficient matrix of the linear system, $\mathbf{x}$ is the vector of variables and $\mathbf{b}$ is the vector of constants on the right hand side of each equation. For example, the matrix equation corresponding to the linear system

$$\begin{cases} -x_1 - x_2 - x_3 & = 2 \\ -x_1 + 2x_2 + 2x_3 & = -3 \\ -x_1 + 2x_2 + x_3 & = 1 \end{cases} \text{ is } \begin{bmatrix} -1 & -1 & -1 \\ -1 & 2 & 2 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

If the coefficient matrix A, as in the previous example, has an inverse, then the linear system always has a unique solution. That is, both sides of the equation $A\mathbf{x} = \mathbf{b}$ can be multiplied on the left by A^{-1} to obtain the solution $\mathbf{x} = A^{-1}\mathbf{b}$. In the above example, since the coefficient matrix is invertible the linear system has a unique solution. That is,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 & -1 & 0 \\ -1 & -2 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 \\ 7 \\ -12 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 7/3 \\ -4 \end{bmatrix}.$$

Every homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has at least one solution, namely the trivial solution, where each component of the vector $\mathbf{x}$ is 0. If in addition, the linear system is an $n \times n$ (square) linear system and A is invertible, then the only solution is the trivial one. That is, the unique solution is $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$. The equivalent statement is that if $A\mathbf{x} = \mathbf{0}$ has two distinct solutions, then the matrix A is not invertible. One additional fact established in the section and that is useful in solving the exercises is that when the linear system $A\mathbf{x} = \mathbf{b}$ has two distinct solutions, then it has infinitely many solutions. That is, every linear linear system has either a unique solution, infinitely many solutions, or is inconsistent.

Solutions to Odd Exercises

1. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 2 & -3 & 1 \\ -1 & -1 & 2 \\ 3 & -2 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$.

5. Let $A = \begin{bmatrix} 4 & 3 & -2 & -3 \\ -3 & -3 & 1 & 0 \\ 2 & -3 & 4 & -4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$.

7. $\begin{cases} 2x - 5y &= 3 \\ 2x + y &= 2 \end{cases}$

9. $\begin{cases} -2y &= 3 \\ 2x - y - z &= 1 \\ 3x - y + 2z &= -1 \end{cases}$

11. $\begin{cases} 2x_1 + 5x_2 - 5x_3 + 3x_4 &= 2 \\ 3x_1 + x_2 - 2x_3 - 4x_4 &= 0 \end{cases}$

13. The solution is $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$.

15. The solution is

$$\mathbf{x} = A^{-1}\mathbf{b} = \mathbf{x} = \begin{bmatrix} 9 \\ -3 \\ -8 \\ 7 \end{bmatrix}.$$

17. The coefficient matrix of the linear system is $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, so that

$$A^{-1} = \frac{1}{(1)(2) - (4)(3)} \begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix} = -\frac{1}{10} \begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix}.$$

Hence, the linear system has the unique solution $\mathbf{x} = \frac{1}{10} \begin{bmatrix} -16 \\ 9 \end{bmatrix}$.

19. If the coefficient matrix is denoted by A, then the unique solution is

$$A^{-1} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -1 \\ -3 & 1 & 0 \\ -8 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \\ 12 \end{bmatrix}.$$

21. If the coefficient matrix is denoted by A, then the unique solution is

$$A^{-1}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}=\begin{bmatrix}1&1&0&0\\-2&-2&1&-2\\0&\frac{1}{3}&-\frac{2}{3}&1\\0&-\frac{1}{3}&-\frac{1}{3}&0\end{bmatrix}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}=\frac{1}{3}\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}.$$

23. a. $\mathbf{x}=\frac{1}{5}\begin{bmatrix}3&1\\-2&1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\frac{1}{5}\begin{bmatrix}7\\-3\end{bmatrix}$ **b.** $\mathbf{x}=\frac{1}{5}\begin{bmatrix}3&1\\-2&1\end{bmatrix}\begin{bmatrix}-3\\2\end{bmatrix}=\frac{1}{5}\begin{bmatrix}-7\\8\end{bmatrix}$

25. The reduced row echelon form of the matrix A is

$$\begin{bmatrix}-1&-4\\3&12\\2&8\end{bmatrix}\longrightarrow\begin{bmatrix}1&4\\0&0\\0&0\end{bmatrix}.$$

hence, the linear system has infinitely many solutions with solution set $S=\left\{\begin{bmatrix}-4t\\t\end{bmatrix}\middle|\, t\in\mathbb{R}\right\}$. A particular nontrivial solution is $x=-4$ and $y=1$.

27. $A=\begin{bmatrix}1&2&1\\1&2&1\\1&2&1\end{bmatrix}$

29. Since $A\mathbf{u}=A\mathbf{v}$ with $\mathbf{u}\neq\mathbf{v}$, then $A(\mathbf{u}-\mathbf{v})=\mathbf{0}$. Hence, the homogeneous linear system $A\mathbf{x}=\mathbf{0}$ has a nontrivial solution, so that A is not invertible.

31. a. Let $A=\begin{bmatrix}2&1\\-1&-1\\3&2\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x\\y\end{bmatrix}$, and $\mathbf{b}=\begin{bmatrix}1\\-2\\-1\end{bmatrix}$. The reduced row echelon form of the augmented matrix is

$$\left[\begin{array}{cc|c}2&1&1\\-1&1&-2\\1&2&-1\end{array}\right]\longrightarrow\left[\begin{array}{cc|c}1&0&1\\0&1&-1\\0&0&0\end{array}\right],$$

so that the solution to the linear system is $\mathbf{x}=\begin{bmatrix}1\\-1\end{bmatrix}$. **b.** $C=\frac{1}{3}\begin{bmatrix}1&-1&0\\1&2&0\end{bmatrix}$

c. The solution to the linear system is also given by $C\mathbf{b}=\frac{1}{3}\begin{bmatrix}1&-1&0\\1&2&0\end{bmatrix}\begin{bmatrix}1\\-2\\-1\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$.

Exercise Set 1.6

The determinant of a square matrix is a number that provides information about the matrix. If the matrix is the coefficient matrix of a linear system, then the determinant gives information about the solutions of the linear system. For a 2×2 matrix $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, then $\det(A)=ad-bc$. Another class of matrices where the finding the determinant requires a simple computation are the triangular matrices. In this case the determinant is the product of the entries on the diagonal. So if $A=(a_{ij})$ is an $n\times n$ matrix, then $\det(A)=a_{11}a_{22}\cdots a_{nn}$. The standard row operations on a matrix can be used to reduce a square matrix to an upper triangular matrix and the affect of a row operation on the determinant can be used to find the determinant of the matrix from the triangular form.

- If two rows are interchanged, then the new determinant is the negative of the original determinant.
- If a row is multiplied by a scalar c, then the new determinant is c times the original determinant.
- If a multiple of one row is added to another, then the new determinant is unchanged.