Exercise Set 1.4

The inverse of a square matrix plays the same role as the reciprocal of a nonzero number for real numbers. The $n \times n$ identity matrix I, with each diagonal entry a 1 and all other entries 0, satisfies $AI = IA = A$ for all $n \times n$ matrices. Then the inverse of an $n \times n$ matrix, when it exists, is unique and is the matrix, denoted by A^{-1}, such that $AA^{-1} = A^{-1}A = I$. In the case of 2×2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has an inverse if and only if } ad - bc \neq 0 \text{ and } A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

A procedure for finding the inverse of an $n \times n$ matrix involves forming the augmented matrix $[A \mid I]$ and then row reducing the $n \times 2n$ matrix. If in the reduction process A is transformed to the identity matrix, then the resulting augmented part of the matrix is the inverse. For example, if

$$A = \begin{bmatrix} -2 & -2 & 1 \\ 1 & -1 & -2 \\ 2 & 1 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -2 & -2 \\ -1 & -1 & 0 \\ 2 & -1 & -1 \end{bmatrix},$$

then A is invertible and B is not since

$$\left[\begin{array}{ccc|ccc} -2 & -2 & 1 & 1 & 0 & 0 \\ 1 & -1 & -2 & 0 & 1 & 0 \\ 2 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} -4 & 3 & -5 \\ 2 & -2 & 3 \\ -3 & 2 & -4 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

but

$$\left[\begin{array}{ccc|ccc} 0 & -2 & -2 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & -4 & -2 \end{array}\right].$$

The inverse of the product of two invertible matrices A and B can be found from the inverses of the individual matrices A^{-1} and B^{-1}. But as in the case of the transpose operation, the order of multiplication is reversed, that is, $(AB)^{-1} = B^{-1}A^{-1}$.

Solutions to Odd Exercises

1. Since $(1)(-1) - (-2)(3) = 5$ and is nonzero, the inverse exists and $A^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix}$.

3. Since $(-2)(-4) - (2)(4) = 0$, then the matrix is not invertible.

5. To determine whether of not the matrix is invertible we row reduce the augmented matrix

$$\left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 0 & 0 & 1 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{(-3)R_1 + R_2 \rightarrow R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & -5 & 4 & 0 & 1 & -3 \\ 0 & 1 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & -5 & 4 & 0 & 1 & -3 \end{array}\right] \xrightarrow{(-2)R_2 + R_1 \rightarrow R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -2 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & -5 & 4 & 0 & 1 & -3 \end{array}\right] \xrightarrow{(5)R_2 + R_3 \rightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -2 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 5 & 1 & -3 \end{array}\right] \xrightarrow{(-1)R_3 \rightarrow R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -2 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -5 & -1 & 3 \end{array}\right] \xrightarrow{(-1)R_3 + R_1 \rightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & -2 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -5 & -1 & 3 \end{array}\right] \xrightarrow{(1)R_3 + R_2 \rightarrow R_2}$$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & -2 \\ 0 & 1 & 0 & -4 & -1 & 3 \\ 0 & 0 & 1 & -5 & -1 & 3 \end{array}\right]$. Since the original matrix has been reduce to the identity matrix, the inverse exists and $A^{-1} = \begin{bmatrix} 3 & 1 & -2 \\ -4 & -1 & 3 \\ -5 & -1 & 3 \end{bmatrix}$.

7. Since the matrix A is row equivalent to the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, the matrix A can not be reduced to the identity and hence is not invertible.

9. $A^{-1} = \begin{bmatrix} 1/3 & -1 & -2 & 1/2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 1/2 \\ 0 & 0 & 0 & -1/2 \end{bmatrix}$

11. $A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 3 & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

13. The matrix A is not invertible.

15. $A^{-1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & -1 & -2 & 1 \\ 1 & -2 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix}$

17. Performing the operations, we have that $AB+A = \begin{bmatrix} 3 & 8 \\ 10 & -10 \end{bmatrix} = A(B+I)$ and $AB+B = \begin{bmatrix} 2 & 9 \\ 6 & -3 \end{bmatrix} = (A+I)B$.

19. Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. **a.** Since $A^2 = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$ and $-2A = \begin{bmatrix} -2 & -4 \\ 4 & -2 \end{bmatrix}$, then $A^2 - 2A + 5I = 0$. **b.** Since $(1)(1) - (2)(-2) = 5$, the inverse exists and $A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}(2I - A)$.
c. If $A^2 - 2A + 5I = 0$, then $A^2 - 2A = -5I$, so that $A\left(\frac{1}{5}(2I - A)\right) = \frac{2}{5}A - \frac{1}{5}A^2 = -\frac{1}{5}(A^2 - 2A) = -\frac{1}{5}(-5I) = I$. Hence $A^{-1} = \frac{1}{5}(2I - A)$.

21. The matrix is row equivalent to $\begin{bmatrix} 1 & \lambda & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 1-2\lambda & 1 \end{bmatrix}$. If $\lambda = -2$, then the second and third rows are identical, so the matrix can not be row reduced to the identity and hence, is not invertible.

23. a. If $\lambda \neq 1$, then the matrix A is invertible.
b. When $\lambda \neq 1$ the inverse matrix is $A^{-1} = \begin{bmatrix} -\frac{1}{\lambda-1} & \frac{\lambda}{\lambda-1} & -\frac{\lambda}{\lambda-1} \\ \frac{1}{\lambda-1} & -\frac{1}{\lambda-1} & \frac{1}{\lambda-1} \\ 0 & 0 & 1 \end{bmatrix}$.

25. The matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are not invertible, but $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is invertible.

27. Using the distributive property of matrix multiplication, we have that

$$\begin{aligned}(A+B)A^{-1}(A-B) &= (AA^{-1} + BA^{-1})(A-B) = (I + BA^{-1})(A-B) \\ &= A - B + B - BA^{-1}B = A - BA^{-1}B.\end{aligned}$$

Similarly, $(A-B)A^{-1}(A+B) = A - BA^{-1}B$.

29. a. If A is invertible and $AB = \mathbf{0}$, then $A^{-1}(AB) = A^{-1}\mathbf{0}$, so that $B = \mathbf{0}$.
b. If A is not invertible, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. Let $\mathbf{x_1}, \ldots, \mathbf{x_n}$ be nonzero solutions of $A\mathbf{x} = \mathbf{0}$ and B the matrix with nth column vector $\mathbf{x_n}$. Then $AB = A\mathbf{x_1} + A\mathbf{x_2} + \cdots + A\mathbf{x_n} = \mathbf{0}$.

31. By the multiplicative property of transpose $(AB)^t = B^tA^t$. Since A and B are symmetric, then $A^t = A$ and $B^t = B$. Hence, $(AB)^t = B^tA^t = BA$. Finally, since $AB = BA$, we have that $(AB)^t = B^tA^t = BA = AB$, so that AB is symmetric.

33. If $AB = BA$, then $B^{-1}AB = A$, so that $B^{-1}A = AB^{-1}$. Then $(AB^{-1})^t = (B^{-1})^t A^t$. Since $(B^{-1})^t B^t = (BB^{-1})^t = I$, we have that $(B^{-1})^t = (B^t)^{-1}$. Finally, $(AB^{-1})^t = (B^t)^{-1}A^t = B^{-1}A = AB^{-1}$ and hence, AB^{-1} is symmetric.

35. Assuming $A^t = A^{-1}$ and $B^t = B^{-1}$, we need to show that $(AB)^t = (AB)^{-1}$. But $(AB)^t = B^tA^t = B^{-1}A^{-1} = (AB)^{-1}$ and hence, AB is orthogonal.

37. a. Using the associative property of matrix multiplication, we have that

$$(ABC)(C^{-1}B^{-1}A^{-1}) = (AB)CC^{-1}(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AA^{-1} = I.$$

b. The proof is by induction on the number of matrices k.
Base Case: When $k = 2$, since $(A_1A_2)^{-1} = A_2^{-1}A_1^{-1}$, the statement holds.
Inductive Hypothesis: Suppose that $(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_1^{-1}$. Then for $k+1$ matrices, we have that $(A_1A_2\cdots A_kA_{k+1})^{-1} = ([A_1A_2\cdots A_k]A_{k+1})^{-1}$. Since $[A_1A_2\cdots A_k]$ and A_{k+1} can be considered as two matrices, by the base case, we have that $([A_1A_2\cdots A_k]A_{k+1})^{-1} = A_{k+1}^{-1}[A_1A_2\cdots A_k]^{-1}$. Finally, by the inductive hypothesis

$$([A_1A_2\cdots A_k]A_{k+1})^{-1} = A_{k+1}^{-1}[A_1A_2\cdots A_k]^{-1} = A_{k+1}^{-1}A_k^{-1}A_{k-1}^{-1}\cdots A_1^{-1}.$$

39. If A is invertible, then the augmented matrix $[A|I]$ can be row reduced to $[I|A^{-1}]$. If A is upper triangular, then only terms on or above the main diagonal can be affected by the reduction process and hence the inverse is upper triangular. Similarly, the inverse for an invertible lower triangle matrix is also lower triangular.

41. a. Expanding the matrix equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, gives $\begin{bmatrix} ax_1+bx_3 & ax_2+bx_4 \\ cx_1+dx_3 & cx_2+dx_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. **b.** From part (a), we have the two linear systems

$$\begin{cases} ax_1 + bx_3 &= 1 \\ cx_1 + dx_3 &= 0 \end{cases} \text{ and } \begin{cases} ax_2 + bx_4 &= 0 \\ cx_2 + dx_4 &= 1 \end{cases}.$$

In the first linear system, multiplying the first equation by d and the second by b and then adding the results gives the equation $(ad - bc)x_1 = d$. Since the assumption is that $ad - bc = 0$, then $d = 0$. Similarly, from the second linear system we conclude that $b = 0$. **c.** From part (b), both $b = 0$ and $d = 0$. Notice that if in addition either $a = 0$ or $c = 0$, then the matrix is not invertible. Also from part (b), we have that $ax_1 = 1, ax_2 = 0, cx_1 = 0$, and $cx_2 = 1$. If a and c are not zero, then these equations are inconsistent and the matrix is not invertible.

Exercise Set 1.5

A linear system can be written as a matrix equation $A\mathbf{x} = \mathbf{b}$, where A is the coefficient matrix of the linear system, $\mathbf{x}$ is the vector of variables and $\mathbf{b}$ is the vector of constants on the right hand side of each equation. For example, the matrix equation corresponding to the linear system

$$\begin{cases} -x_1 - x_2 - x_3 &= 2 \\ -x_1 + 2x_2 + 2x_3 &= -3 \\ -x_1 + 2x_2 + x_3 &= 1 \end{cases} \text{ is } \begin{bmatrix} -1 & -1 & -1 \\ -1 & 2 & 2 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

If the coefficient matrix A, as in the previous example, has an inverse, then the linear system always has a unique solution. That is, both sides of the equation $A\mathbf{x} = \mathbf{b}$ can be multiplied on the left by A^{-1} to obtain the solution $\mathbf{x} = A^{-1}\mathbf{b}$. In the above example, since the coefficient matrix is invertible the linear system has