Here we used that two matrices are equal if and only if corresponding components are equal. Some of the exercises involve showing some matrix or combination of matrices is symmetric. For example, to show that the product of two matrices AB is symmetric, requires showing that $(AB)^t = AB$.

Solutions to Odd Exercises

1. Since addition of matrices is defined componentwise, we have that

$$A+B=\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}+\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix}=\begin{bmatrix} 2-1 & -3+3 \\ 4-2 & 1+5 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 2 & 6 \end{bmatrix}.$$

Also, since addition of real numbers is commutative, $A+B=B+A$.

3. To evaluate the matrix expression $(A+B)+C$ requires we first add $A+B$ and then add C to the result. On the other hand to evaluate $A+(B+C)$ we first evaluate $B+C$ and then add A. Since addition of real numbers is associative the two results are the same, that is $(A+B)+C=\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}=A+(B+C)$.

5. Since a scalar times a matrix multiples each entry of the matrix by the real number, we have that

$$(A-B)+C=\begin{bmatrix} -7 & -3 & 9 \\ 0 & 5 & 6 \\ 1 & -2 & 10 \end{bmatrix} \quad \text{and} \quad 2A+B=\begin{bmatrix} -7 & 3 & 9 \\ -3 & 10 & 6 \\ 2 & 2 & 11 \end{bmatrix}.$$

7. The products are $AB=\begin{bmatrix} 7 & -2 \\ 0 & -8 \end{bmatrix}$ and $BA=\begin{bmatrix} 6 & 2 \\ 7 & -7 \end{bmatrix}$. Notice that, A and B are examples of matrices that do not commute, that is, the order of multiplication can not be reversed.

9. $AB=\begin{bmatrix} -9 & 4 \\ -13 & 7 \end{bmatrix}$

11. $AB=\begin{bmatrix} 5 & -6 & 4 \\ 3 & 6 & -18 \\ 5 & -7 & 6 \end{bmatrix}$

13. First, adding the matrices B and C gives

$$\begin{aligned} A(B+C) &= \begin{bmatrix} -2 & -3 \\ 3 & 0 \end{bmatrix}\left(\begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}+\begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}\right) \\ &= \begin{bmatrix} -2 & -3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ -3 & -1 \end{bmatrix}=\begin{bmatrix} (-2)(4)+(-3)(-3) & (-2)(0)+(-3)(-1) \\ (3)(4)+(0)(-3) & (3)(0)+(0)(-1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 12 & 0 \end{bmatrix}. \end{aligned}$$

15. $2A(B-3C)=\begin{bmatrix} 10 & -18 \\ -24 & 0 \end{bmatrix}$

17. To find the transpose of a matrix the rows and columns are reversed. So A^t and B^t are 3×2 matrices and the operation is defined. The result is $2A^t-B^t=\begin{bmatrix} 7 & 5 \\ -1 & 3 \\ -3 & -2 \end{bmatrix}$.

19. Since A is 2×3 and B^t is 3×2, then the product AB^t is defined with $AB^t=\begin{bmatrix} -7 & -4 \\ -5 & 1 \end{bmatrix}$

21. $(A^t+B^t)C=\begin{bmatrix} -1 & 7 \\ 6 & 8 \\ 4 & 12 \end{bmatrix}$

23. $(A^tC)B=\begin{bmatrix} 0 & 20 & 15 \\ 0 & 0 & 0 \\ -18 & -22 & -15 \end{bmatrix}$

25. $AB=AC=\begin{bmatrix} -5 & -1 \\ 5 & 1 \end{bmatrix}$

27. The product

$$A^2 = AA = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a^2 & ab+bc \\ 0 & c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

if and only if $a^2 = 1, c^2 = 1$, and $ab + bc = b(a + c) = 0$. That is, $a = \pm 1, c = \pm 1$, and $b(a + c) = 0$, so that A has one of the forms $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}$, or $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

29. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and neither of the matrices are the zero matrix. Notice that, this can not happen with real numbers. That is, if the product of two real numbers is zero, then at least one of the numbers must be zero.

31. The product

$$\begin{bmatrix} 1 & 2 \\ a & 0 \end{bmatrix}\begin{bmatrix} 3 & b \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -5 & b+2 \\ 3a & ab \end{bmatrix}$$

will equal $\begin{bmatrix} -5 & 6 \\ 12 & 16 \end{bmatrix}$ if and only $b + 2 = 6, 3a = 12$, and $ab = 16$. That is, $a = b = 4$.

33. Several powers of the matrix A are given by

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can see that if n is even, then A^n is the identity matrix, so in particular $A^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Notice also that, if n is odd, then $A^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

35. We can first rewrite the expression A^2B as $A^2B = AAB$. Since $AB = BA$, then $A^2B = AAB = ABA = BAA = BA^2$.

37. Multiplication of A times the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ gives the first column vector of the matrix A. Then $A\mathbf{x} = \mathbf{0}$ forces the first column vector of A to be the zero vector. Then let $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ and so on, to show that each column vector of A is the zero vector. Hence, A is the zero matrix.

39. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so that $A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Then

$$AA^t = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

if and only if $a^2+b^2 = 0, c^2+d^2 = 0$, and $ac+bd = 0$. The only solution to these equations is $a = b = c = d = 0$, so the only matrix that satisfies $AA^t = \mathbf{0}$ is the 2×2 zero matrix.

41. If A is an $m \times n$ matrix, then A^t is an $n \times m$ matrix, so that AA^t and A^tA are both defined with AA^t being an $m \times m$ matrix and A^tA an $n \times n$ matrix. Since $(AA^t)^t = (A^t)^tA^t = AA^t$, then the matrix AA^t is symmetric. Similarly, $(A^tA)^t = A^t(A^t)^t = A^tA$, so that A^tA is also symmetric.

43. Let $A = (a_{ij})$ be an $n \times n$ matrix. If $A^t = -A$, then the diagonal entries satisfy $a_{ii} = -a_{ii}$ and hence $a_{ii} = 0$ for each i.