Solutions to All Odd–Numbered Exercises

1 Systems of Linear Equations and Matrices

Exercise Set 1.1

In Section 1.1 of the text, Gaussian Elimination is used to solve a linear system. This procedure utilizes three operations that when applied to a linear system result in a new system that is equivalent to the original. Equivalent means that the linear systems have the same solutions. The three operations are:

- Interchange two equations.
- Multiply any equation by a nonzero constant.
- Add a multiple of one equation to another.

When used judiciously these three operations allow us to reduce a linear system to a triangular linear system, which can be solved. A linear system is consistent if there is at least one solution and is inconsistent if there are no solutions. Every linear system has either a unique solution, infinitely many solutions or no solutions. For example, the triangular linear systems

$$\begin{cases} x_1 - x_2 + x_3 &= 2 \\ \quad\;\; x_2 - 2x_3 &= -1 \\ \qquad\quad\; x_3 &= 2 \end{cases}, \quad \begin{cases} x_1 - 2x_2 + x_3 &= 2 \\ \quad\;\; -x_2 + 2x_3 &= -3 \\ \qquad\qquad\qquad \end{cases}, \quad \begin{cases} 2x_1 \qquad\quad + x_3 &= 1 \\ \qquad\; x_2 \qquad - x_3 &= 2 \\ \qquad\qquad\quad\; 0 &= 4 \end{cases}$$

have a unique solution, infinitely many solutions, and no solutions, respectively. In the second linear system, the variable x_3 is a free variable, and once assigned any real number the values of x_1 and x_2 are determined. In this way the linear system has infinitely many solutions. If a linear system has the same form as the second system, but also has the additional equation $0 = 0$, then the linear system will still have free variables. The third system is inconsistent since the last equation $0 = 4$ is impossible. In some cases, the conditions on the right hand side of a linear system are not specified. Consider for example, the linear system

$$\begin{cases} -x_1 - x_2 \qquad\quad &= a \\ 2x_1 + 2x_2 + x_3 &= b \\ \qquad\qquad\quad 2x_3 &= c \end{cases} \xrightarrow{\text{which is equivalent to}} \begin{cases} -x_1 - x_2 \qquad\quad &= a \\ \qquad\qquad\quad x_3 &= b + 2a \\ \qquad\qquad\quad 0 &= c - 2b - 4a \end{cases}.$$

This linear system is consistent only for values a, b and c such that $c - 2b - 4a = 0$.

Solutions to Odd Exercises

1. Applying the given operations we obtain the equivalent triangular system

$$\begin{cases} x_1 - x_2 - 2x_3 &= 3 \\ -x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 - 2x_2 - 2x_3 &= -2 \end{cases} \xrightarrow{E_1 + E_2 \to E_2} \begin{cases} x_1 - x_2 - 2x_3 &= 3 \\ \qquad\quad x_2 + x_3 &= 4 \\ 2x_1 - 2x_2 - 2x_3 &= -2 \end{cases} \xrightarrow{(-2)E_1 + E_3 \to E_3}$$

$$\begin{cases} x_1 - x_2 - 2x_3 &= 3 \\ \quad x_2 + x_3 &= 4 \\ \qquad 2x_3 &= -8 \end{cases}$$

. Using back substitution, the linear system has the unique solution

$$x_1 = 3, x_2 = 8, x_3 = -4.$$

3. Applying the given operations we obtain the equivalent triangular system

$$\begin{cases} x_1 & & + 3x_4 &= 2 \\ x_1 + x_2 & & + 4x_4 &= 3 \\ 2x_1 & + x_3 & + 8x_4 &= 3 \\ x_1 + x_2 + x_3 & & + 6x_4 &= 2 \end{cases} \xrightarrow{(-1)E_1 + E_2 \to E_2} \begin{cases} x_1 & & + 3x_4 &= 2 \\ & x_2 & + x_4 &= 1 \\ 2x_1 & + x_3 & + 8x_4 &= 3 \\ x_1 + x_2 + x_3 & & + 6x_4 &= 2 \end{cases}$$

$$\xrightarrow{(-2)E_1 + E_3 \to E_3} \begin{cases} x_1 & & + 3x_4 &= 2 \\ & x_2 & + x_4 &= 1 \\ & +x_3 & + 2x_4 &= -1 \\ x_1 + x_2 + x_3 & & + 6x_4 &= 2 \end{cases} \xrightarrow{(-1)E_1 + E_4 \to E_4}$$

$$\begin{cases} x_1 & & + 3x_4 &= 2 \\ & x_2 & + x_4 &= 1 \\ & +x_3 & + 2x_4 &= -1 \\ & x_2 + x_3 & + 3x_4 &= 0 \end{cases} \xrightarrow{(-1)E_2 + E_4 \to E_4} \begin{cases} x_1 & & + 3x_4 &= 2 \\ & x_2 & + x_4 &= 1 \\ & x_3 & + 2x_4 &= -1 \\ & x_3 & + 2x_4 &= -1 \end{cases}$$

$$\xrightarrow{(-1)E_3 + E_4 \to E_4} \begin{cases} x_1 & & + 3x_4 &= 2 \\ & x_2 & + x_4 &= 1 \\ & x_3 & + 2x_4 &= -1 \\ & & 0 &= 0 \end{cases}.$$

The final triangular linear system has more variables than equations, that is, there is a free variable. As a result there are infinitely many solutions. Specifically, using back substitution, the solutions are given by $x_1 = 2 - 3x_4, x_2 = 1 - x_4, x_3 = -1 - 2x_4$, $x_4 \in \mathbb{R}$.

5. The second equation gives immediately that $x = 0$. Substituting the value $x = 0$ into the first equation we have that $y = -\frac{2}{3}$. Hence, the linear system has the unique solution $x = 0, y = -\frac{2}{3}$.

7. The first equation gives $x = 1$ and substituting this value in the second equation, we have $y = 0$. Hence the linear system has the unique solution $x = 1, y = 0$.

9. Notice that the first equation is three times the second and hence, the equations have the same solutions. Since each equation has infinitely many solutions the linear system has infinitely many solutions with solution set $S = \left\{ \left(\frac{2t+4}{3}, t\right) \middle| \ t \in \mathbb{R} \right\}$.

11. The operations $E_1 \leftrightarrow E_3, E_1 + E_2 \to E_2$, $3E_1 + E_3 \to E_3$ and $-\frac{8}{5}E_2 + E_3 \to E_3$, reduce the linear system to the equivalent triangular system

$$\begin{cases} x - 2y + z &= -2 \\ \quad -5y + 2z &= -5 \\ \qquad \frac{9}{5}z &= 0 \end{cases}.$$

The unique solution is $x = 0, y = 1, z = 0$.

13. The operations $E_1 \leftrightarrow E_2, 2E_1 + E_2 \to E_2$, $-3E_1 + E_3 \to E_3$ and $E_2 + E_3 \to E_3$, reduce the linear system to the equivalent triangular system

$$\begin{cases} x & + 5z &= -1 \\ \quad -2y & + 12z &= -1 \\ \qquad & 0 &= 0 \end{cases}.$$

The linear system has infinitely many solutions with solution set $S = \left\{ \left(-1 - 5t, 6t + \frac{1}{2}, t\right) \middle| \ t \in \mathbb{R} \right\}$.

15. Adding the two equations yields $6x_1 + 6x_3 = 4$, so that $x_1 = \frac{2}{3} - x_3$. Substituting this value in the first equation gives $x_2 = -\frac{1}{2}$. The linear system has infinitely many solutions with solution set $S = \left\{ \left(-t + \frac{2}{3}, -\frac{1}{2}, t\right) \middle| \ t \in \mathbb{R} \right\}$.

17. The operation $2E_1 + E_2 \to E_2$ gives the equation $-3x_2 - 3x_3 - 4x_4 = -9$. Hence, the linear system has two free variables, x_3 and x_4. The two parameter set of solutions is $S = \left\{ \left(3 - \frac{5}{3}t, -s - \frac{4}{3}t + 3, s, t\right) \middle| \ s, t \in \mathbb{R} \right\}$.

19. The operation $-2E_1 + E_2 \to E_2$ gives $x = b - 2a$. Then $y = a + 2x = a + 2(b - 2a) = 2b - 3a$, so that the unique solution is $x = -2a + b, y = -3a + 2b$.

21. The linear system is equivalent to the triangular linear system

$$\begin{cases} -x \qquad\quad - z &= b \\ \qquad y \qquad\quad &= a + 3b \\ \qquad\qquad z &= c - 7b - 2a \end{cases},$$

which has the unique solution $x = 2a + 6b - c, y = a + 3b, z = -2a - 7b + c$.

23. Since the operation $2E_1 + E_2 \to E_2$ gives the equation $0 = 2a + 2$, then the linear system is consistent for $a = -1$.

25. Since the operation $2E_1 + E_2 \to E_2$ gives the equation $0 = a + b$, then the linear system is consistent for $b = -a$.

27. The linear system is equivalent to the triangular linear system

$$\begin{cases} x - 2y + 4z &= a \\ \quad 5y - 9z &= -2a + b \\ \qquad\quad 0 &= c - a - b \end{cases}$$

and hence, is consistent for all a, b, and c such that $c - a - b = 0$.

29. The operation $-2E_1 + E_2 \to E_2$ gives the equivalent linear system

$$\begin{cases} x + y &= -2 \\ (a-2)y &= 7 \end{cases}.$$

Hence, if $a = 2$, the linear system is inconsistent.

31. The operation $-3E_1 + E_2 \to E_2$ gives the equivalent linear system

$$\begin{cases} x - y &= 2 \\ 0 &= a - 6 \end{cases}.$$

Hence, the linear system is inconsistent for all $a \neq 6$.

33. To find the parabola $y = ax^2 + bx + c$ that passes through the specified points we solve the linear system

$$\begin{cases} c &= 0.25 \\ a + b + c &= -1.75 \\ a - b + c &= 4.25 \end{cases}.$$

The unique solution is $a = 1, b = -3$, and $c = \frac{1}{4}$, so the parabola is $y = x^2 - 3x + \frac{1}{4} = \left(x - \frac{3}{2}\right)^2 - 2$. The vertex of the parabola is the point $\left(\frac{3}{2}, -2\right)$.

35. To find the parabola $y = ax^2 + bx + c$ that passes through the specified points we solve the linear system

$$\begin{cases} (-0.5)^2a - (0.5)b + c &= -3.25 \\ a \quad + \quad b + c &= 2 \\ (2.3)^2a \quad + (2.3)b + c &= 2.91 \end{cases}.$$

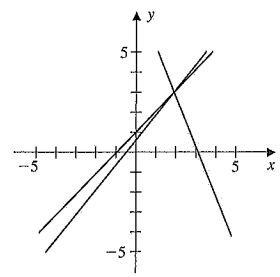
The unique solution is $a = -1, b = -4$, and $c = -1$, so the parabola is $y = -x^2 + 4x - 1 = -(x-2)^2 + 3$. The vertex of the parabola is the point $(2, 3)$.

37. a. The point of intersection of the three lines can be found by solving the linear system

$$\begin{cases} -x + y &= 1 \\ -6x + 5y &= 3 \\ 12x + 5y &= 39 \end{cases}.$$

This linear system has the unique solution $(2, 3)$.

b.



39. a. The linear system $\begin{cases} x + y &= 2 \\ x - y &= 0 \end{cases}$ has the unique solution $x = 1$ and $y = 1$. Notice that the two lines have different slopes.

b. The linear system $\begin{cases} x + y &= 1 \\ 2x + 2y &= 2 \end{cases}$ has infinitely many solutions given by the one parameter set $S = \{(1 - t, t) \mid t \in \mathbb{R}\}$. Notice that the second equation is twice the first and the equations represent the same line.

c. The linear system $\begin{cases} x + y &= 2 \\ 3x + 3y &= -6 \end{cases}$ is inconsistent.

41. a. $S = \{(3 - 2s - t, 2 + s - 2t, s, t) \mid s, t \in \mathbb{R}\}$ **b.** $S = \{(7 - 2s - 5t, s, -2 + s + 2t, t) \mid s, t \in \mathbb{R}\}$

43. Applying $kE_1 \to E_1, 9E_2 \to E_2$, and $-E_1 + E_2 \to E_2$ gives the equivalent linear system

$$\begin{cases} 9kx + \quad k^2y &= 9k \\ \quad (9 - k^2)y &= -27 - 9k \end{cases}.$$

Whether the linear system is consistent or inconsistent can now be determined by examining the second equation.

a. If $k = 3$, the second equation becomes $0 = -54$, so the linear system is inconsistent. **b.** If $k = -3$, then the second equation becomes $0 = 0$, so the linear system has infinitely many solutions. **c.** If $k \neq \pm 3$, then the linear system has a unique solution.

Exercise Set 1.2

Matrices are used to provide an alternative way to represent a linear system. Reducing a linear system to triangular form is then equivalent to row reducing the augmented matrix corresponding to the linear system to a triangular matrix. For example, the augmented matrix for the linear system

$$\begin{cases} -x_1 - x_2 - x_3 - 2x_4 &= 1 \\ 2x_1 + 2x_2 + x_3 - 2x_4 &= 2 \\ x_1 - 2x_2 + x_3 + 2x_4 &= -2 \end{cases} \quad \text{is} \quad \left[\begin{array}{cccc|c} -1 & -1 & -1 & -2 & 1 \\ 2 & 2 & 1 & -2 & 2 \\ 1 & -2 & 1 & 2 & -2 \end{array}\right].$$

The coefficient matrix is the 3×4 matrix consisting of the coefficients of each variable, that is, the augmented