#32 $c_1 v_1 + c_2 v_2 + c_3 v_3 = v$.

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & a \\ 0 & -2 & -1 & b \\ 2 & 8 & 3 & c \end{array}\right] \xrightarrow{2R_1+R_3} \left[\begin{array}{ccc|c} -1 & 1 & 1 & a \\ 0 & -2 & -1 & b \\ 0 & 10 & 5 & 2a+c \end{array}\right] \xrightarrow{-5R_2+R_3}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & a \\ 0 & -2 & -1 & b \\ 0 & 0 & 0 & 5b+2a+c \end{array}\right]$$

this system has a solution for $5b+2a+c=0$ (is consistent).

$$v = \begin{bmatrix} a \\ b \\ 5b+2a+c \end{bmatrix}$$

Review

Section 1.1

#44 (p. 14)

$$\begin{cases} kx + y + z = 0 \\ x + ky + z = 0 \\ x + y + kz = 0 \end{cases}$$

a) $$\left[\begin{array}{ccc|c} k & 1 & 1 & 0 \\ 1 & k & 1 & 0 \\ 1 & 1 & k & 0 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|c} k & 1 & 1 & 0 \\ 1 & k & 1 & 0 \\ 0 & -k+1 & k-1 & 0 \end{array}\right] \xrightarrow{-\frac{1}{k}R_1+R_2} \left[\begin{array}{ccc|c} k & 1 & 1 & 0 \\ 0 & k-\frac{1}{k} & 1-\frac{1}{k} & 0 \\ 0 & 1-k & k-1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_1+R_3} \left[\begin{array}{ccc|c} k & 1 & 1 & 0 \\ 0 & k-\frac{1}{k} & 1-\frac{1}{k} & 0 \\ 0 & -k & k-2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} k & 1 & 1 & 0 \\ 0 & k^2-1 & k-1 & 0 \\ 0 & -k & k-2 & 0 \end{array}\right] \xrightarrow{\frac{k}{k^2-1}R_2+R_3} \left[\begin{array}{ccc|c} k & 1 & 1 & 0 \\ 0 & k^2-1 & k-1 & 0 \\ 0 & 0 & 2-k-k^2 & 0 \end{array}\right]$$

$(2k-k^2)z = 0$. $\rightarrow$ unique solution: $2-k-k^2 \neq 0$

$k \neq -2$; $k \neq 1$

also: $(k^2-1)y = 0$

$\Rightarrow k^2-1 \neq 0$ $k \neq -1$ check answer.

b) together, $k = 2$. c) $k = 1$

b) $\begin{cases} 2-k-k^2 = 0 \\ \text{but } (k-1) \neq 0 \qquad k \neq 1 \end{cases}$

so $k = -2$.

c) $\begin{cases} 2-k-k^2 & k=1 \\ & k=-2 \\ \text{and} & \\ k-1=0 & k=1 \end{cases}$

So $k=1$.

Exercices:

1. $\left[\begin{array}{ccc|c} a & 0 & 1 & 2 \\ a & a & 4 & -4 \\ 0 & a-1 & 2 & 6 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|c} a & 0 & 1 & 2 \\ 0 & a & 3 & -6 \\ 0 & a-1 & 2 & b \end{array}\right]$

$\xrightarrow{-\frac{(a-1)}{a}R_2+R_3} \left[\begin{array}{ccc|c} a & 0 & 1 & 2 \\ 0 & a & 3 & -6 \\ 0 & 0 & -\frac{3(a-1)}{a}+b & \frac{6(a-1)}{a}+b \end{array}\right]$ Cond: $a \neq 0$

$ax + z = 2$

$ay + 3z = -6$

$\left(-\frac{3(a-1)}{a}+2\right) z = \frac{6(a-1)}{a} + b$

$\left(\frac{-3(a-1)+2a}{a}\right) z = \frac{6a-6+ab}{a}$

$-a+3 \quad z = 6a-6+ab.$

unique solution: $-a+3 \neq 0 \qquad a \neq 3$

infinitely many solutions $\begin{cases} -a+3=0 \\ 6a-6+ab=0 \end{cases}$ $\quad a=3$

$3b=-12$

$b=-4$

no solution: $\begin{cases} 6a - 6 + ab \neq 0 \\ -a + 3 \cancel{\neq} = 0 \end{cases}$

$a = 3 \quad b \neq -4.$

But we need to see what happens for $a = 0$

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & -6 \\ 0 & -1 & 2 & 6 \end{bmatrix}$$

3. a) $\begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b) $A^2 = 0 \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$a^2 + bc = 0$

$ab + bd = 0$

$ca + dc = 0$

$cb + d^2 = 0$

$a^2 = d^2$

$a = d$ ou $a = -d$

$a = 0$ or $b = 0$

$d = 0$

$b = 0$ $c = 0$

$a = 0$, $b = 0$, $c =$ anything, $d = 0$

$a = 0$, b anything, $c = 0$, $d = 0$

$a = 0$

$d = 0$

c anything

$a^2 + bc = 0$

5) $\left(I + (2A)^{-1}\right)^{-1} = \begin{pmatrix} 1 & -2 \\ -3 & 5 \end{pmatrix}$

Find A.

$$I + (2A)^{-1} = \frac{1}{-1}\begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}$$

$$(2A)^{-1} = \begin{pmatrix} -5 & -2 \\ -3 & -1 \end{pmatrix} - I$$

$$= \begin{pmatrix} -6 & -2 \\ -3 & -2 \end{pmatrix}$$

$$2A = \frac{1}{6}\begin{pmatrix} -2 & 2 \\ 3 & -6 \end{pmatrix}$$

$$2A = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix}$$

8) $J_n = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ & \dots & & \end{bmatrix}$

$n > 1$

$$(I - J_n)^{-1} = I - \frac{1}{n-1} J_n$$

$$(I - J_n)\left(I - \frac{1}{n-1} J_n\right) = I - \frac{1}{n-1} J_n - J_n + \frac{1}{n-1} J_n^2$$

$$= I - \frac{(1+n-1)}{n-1} J_n + \frac{1}{n-1} J_n^2$$

$$= I - \frac{n}{n-1} J_n + \frac{n}{n-1} J_n = I$$

$$J_n^2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} n & n & n & \cdots \end{bmatrix}$$

$$= n\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= n\,[J_n].$$

likewise $\left(I - \frac{1}{n-1} J_n\right)(I - J_n) = I.$

Def. inverse : B inverse of A if $AB = BA = I$.

theorem : 1) B is inverse of A if $AB = I$.
2) B is inverse of A if $BA = I$.

9) a) If $(A^t A)$ is invertible then A is invertible.

True since.
$A^t A$ is inv. $\det(A^t A) \neq 0$

But $\det(A^t A) = \det(A^t)\det(A) = \det(A)\det(A) \neq 0$
so $\det(A) \neq 0$
so A is invertible

b) $\det(A+B) = \det(A) + \det(B)$. False

c) A is invertible
$\Rightarrow$ $Ax = x$ has exactly one solution. False.
$Ax - x = 0$
$(A - I)x = 0$ has only trivial solution.
$A - I$ is invertible

$$\overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} - \overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \overset{A-I}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$$

§1.4. 21) $\begin{bmatrix} 1 & \lambda & 0 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ Method 1: RREF. Method 2: Determinants.

§1.4. 32. A, B inv. sym. $AB = BA$.

Show $A^{-1}B$ is sym.

$\cancel{(A+B)^{-1} = A^{-1} + B^{-1}}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$(A^{-1}B)^t = B^t (A^{-1})^t$

$\quad = B (A^t)^{-1} = BA^{-1}$

$ABA^{-1} = BAA^{-1}$

$A^{-1}ABA^{-1} = A^{-1}B$

$BA^{-1} = A^{-1}B$

so $(A^{-1}B)^t = BA^{-1} = A^{-1}B$

so $A^{-1}B$ is symmetric

$(AB)^t = \cancel{A^tB^t}$
$= B^tA^t$

$(AB)^{-1} = B^{-1}A^{-1}$
$= \cancel{A^{-1}B^{-1}}$

35. Def. A is orthogonal iff $= A^t = A^{-1}$.

Given: A, B are orthogonal.

Want: AB is orthogonal

$A^t = A^{-1}$

$B^t = B^{-1}$

want: $(AB)^t = (AB)^{-1}$

$(AB)^t = B^tA^t$

$= B^{-1}A^{-1}$

$= (AB)^{-1}$

theorem: AB invertible $\iff$ A is inv. and B is invertible

§1.6 n°52. A is n×n.

A is skew symmetric

Def. iff $A^t = -A$.

Given: n odd positive integer.

A is skew symmetric of size n×n.

want: A is not invertible.

$$A^t = -A.$$

$$\det(A^t) = \det(-A)$$

$$\det(A) = (-1)^n \det(A)$$

$$\det(A) = -\det(A)$$

$$\det(A) = 0.$$

A is not invertible.

#36.

$$\begin{cases} x + 3y - 2z = -1 \\ 2x + 5y + z = 2 \\ 2x + 6y - 4z = -2. \end{cases}$$

a) $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & 1 \\ 2 & 6 & -4 \end{bmatrix}.$

b) $\det(A)$

c) $\det(A) \neq 0 \Rightarrow A$ inv $\iff$ unique solution.

ex : $\det(A) = 3$
$\det(B) = 4$.

$A \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $C = \begin{bmatrix} a & b & c \\ 3a+d & 3b+e & 3c+f \\ 2g & 2h & 2i \end{bmatrix}$

$$\det((2A)^{-1} BA) = \det((2A)^{-1}) \det(B) \det(A)$$

$$= \frac{1}{\det(2A)} \det(B) \det(A)$$

$$= \frac{1}{2^3 \det(A)} \cdot \det(B) \det(A)$$

$$= \frac{4}{2^3} = \frac{1}{2}.$$

$$\det(CA^2) = 2 \cancel{\det} \times 3 \times 3 \times 3$$

$$\det(C) = 2\det(A) = 2 \times 3$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \frac{1}{k-1} \begin{vmatrix} ka_1 - a_1 & a_2 & ma_2 + a_3 \\ kb_1 - b_1 & b_2 & mb_2 + b_3 \\ kc_1 - c_1 & c_2 & mc_2 + c_3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 123 \\ 4 & 5 & 456 \\ 7 & 8 & 789 \end{vmatrix}$$

$123 = 100 \times 1 + 10 \times 2 + 3$

- $A^3 + 4AA^T - 2A - 7I = 0$