Topology Final Exam

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Date: January 15, 2014

Instructions: Print your *name* and *Topology Final Exam* in the upper right corner of the first page. Also be sure to put your name in the upper right corner of *every* page you intend to turn in. The value of each problem is indicated in **bold**.

Throughout the exam, $I := [0, 1]$ denotes the closed unit interval with the usual topology and $\omega := \{1, 2, \dots\}$ (with the discrete topology).

The first problem is like the first problem on Midterm 1 (the "forgotten midterm"). We will consider the following "constructions" with spaces:

(1) subspace
(2) closed subspace
(3) *finite* product
(4) *finite* coproduct

and the following "properties" of topological spaces:

(1) compact
(2) Hausdorff
(3) second countable (has a countable basis)
(4) metrizable
(5) locally compact
(6) path connected
(7) simply connected

(1) **(5)** Construct a 4×7 matrix by setting the (i, j) entry of the matrix equal to 1 if every i of a space (or "spaces," as appropriate) with property j also has property j—otherwise set the (i, j) entry to zero. For example, the $(3, 2)$ entry of this matrix is 1 if every *finite product* of *Hausdorff* spaces is *Hausdorff*—otherwise the $(3, 2)$ entry is zero. No explanation or proofs are necessary—the matrix is your entire answer.
Solution:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

I didn't count the question about a finite coproduct of simply connected spaces, because I decided I was not clear about the convention that a simply connected space *is required to be path connected* (this is the standard convention). At most one wrong was a 5, two or three wrong was a 4, and so forth.

(2) **(4)** Give $I^\omega := \{x = (x_1, x_2, \dots) : x_1, x_2, \dots \in I\}$ the product topology. Give

$$X := \{x \in I^\omega : x_n = 0 \text{ for all sufficiently large } n\}$$

the subspace topology. Is X compact? Justify your answer.
Solution: Since I^ω is compact Hausdorff, X is compact iff it is closed in I^ω (a closed subspace of a compact space is compact, and a compact space is closed in any Hausdorff space containing it). If you made this observation you got at least 2 points. To show that X is not compact, we will show that it isn't closed in I^ω. In fact it is far from being closed... it is even dense in I^ω (though it is clearly not equal to I^ω). Indeed, a non-empty basic open subset of I^ω takes the form

$$U = U_1 \times \cdots \times U_n \times I \times I \times \cdots$$

where $U_1, \dots, U_n$ are non-empty open subsets of I. Take any points

$$x_1 \in U_1, \dots, x_n \in U_n.$$

Then $x := (x_1, \dots, x_n, 0, 0, \dots)$ is in $U \cap X$.

(3) **(3)** Is there a linear order $<$ on $\mathbb{R}^2$ such the topology on $\mathbb{R}^2$ induced by $<$ coincides with the usual topology? Justify your answer.
Solution: No. Say $<$ is a linear order on $\mathbb{R}^2$. Take any three points $x, y, z \in \mathbb{R}^2$. After renaming them, we can assume $x < y < z$. Then

$$\mathbb{R}^2 \setminus \{y\} = \{p \in \mathbb{R}^2 : p < y\} \coprod \{q \in \mathbb{R}^2 : y < q\}$$

shows that $\mathbb{R}^2 \setminus \{y\}$ is disconnected in the order topology, which is not true in the usual topology.

(4) **(3)** Fill in the blanks:

$$\pi_1(S^1 \times S^1) \cong \_ \quad \pi_1(S^2) \cong \_ \quad \pi_1(\mathbb{R}^2) \cong \_ \quad \pi_1(S^2/\{\pm 1\}) \cong \_$$

Solution: The blanks are $\mathbb{Z}^2$, $\{1\}$, $\{1\}$, $\{\pm 1\}$, as you learned on the previous midterm.

(5) **(4)** Prove the following statements about covering spaces:
(a) If $f : \tilde{X} \to X$ and $g : \tilde{Y} \to Y$ are covering spaces, then

$$f \times g : \tilde{X} \times \tilde{Y} \to X \times Y$$

is a covering space.
(b) If $f : X \to Y$ and $g : Y \to Z$ are covering spaces, then $gf : X \to Z$ is a covering space.

Error: I made an error here: It is necessary to assume that g is finite-to-one in (b). Obviously I did not grade you on (b); I just made this a two-point problem. I think everyone got the two points.

Here is an example that shows you need to assume g is finite-to-one: Take $Z = S^1$, take $Y = \coprod_{n=1}^\infty S^1$, g the trivial covering space map, $X = \coprod_{n=1}^\infty S^1$, and f the coproduct of the n^{th} power maps $f_n : S^1 \to S^1$. The trouble is that there is no single neighborhood U of, say, $1 \in S^1 \subseteq \mathbb{C}^*$ which is good for *all* of the f_n.

Solution: For (a), consider a point $(x, y) \in X \times Y$. Pick a neighborhood U (resp. V) of x (resp. y) in X (resp. Y) good for f (resp. g). Then $f^{-1}(U) = \coprod_i U_i$ is a disjoint union of opens U_i, each mapping homeomorphically to U via (the restriction of) f, and $g^{-1}(V) = \coprod_j V_j$ is a similar disjoint union. Then

$$\begin{aligned}(f \times g)^{-1}(V \times U) &= f^{-1}(V) \times g^{-1}(U) \\ &= \coprod_{i,j} U_i \times V_j\end{aligned}$$

and each $U_i \times V_j$ maps homeomorphically to $U \times V$ via (the restriction of) $f \times g$, hence $U \times V$ is good for $f \times g$.

For (b), pick any $z \in Z$. Pick a neighborhood U of z in Z good for g, so $g^{-1}(U) = \coprod_{i=1}^n U_i$, with each map $g|U_i : U_i \to U$ a homeomorphism (there can only be finitely many U_i by the assumption that g is finite-to-one). Let y_i be the unique element of $g^{-1}(z) \cap U_i$. Pick a good neighborhood V_i of y_i for f, so that $f^{-1}(V_i) = \coprod_{j \in I_j} V_{i,j}$, with each map $f|V_{i,j} \to V_i$ a homeomorphism. Replacing V_i with $V_i \cap g^{-1}(U)$ if necessary, we can assume $V_i \subseteq g^{-1}(U)$. We noted on some homework that a covering map is open, so $W := g(V_1) \cap \cdots \cap g(V_n)$ is a neighborhood of z, which is easily seen to be good.

Do **only two** of these last three problems:

(6) **(5)** Let $X = (X, \tau)$ be a topological space. Let τ_k be the family of subsets U of X such that $U \cap K$ is open in K for every *compact* subspace $K \subseteq X$. Clearly $\tau \subseteq \tau_k$.

(a) Prove that τ_k is a topology (the *compactly generated topology*, or *k-topology*, from the German *kompakt*).

(b) Prove that $\tau = \tau_k$ when X is first countable. *Hint:* For such an X there is a useful criterion for determining when a subset $Z \subseteq X$ is closed.

(It is not so easy to give an example where $\tau \neq \tau_k$! Be thankful that there is no part (c)!)

Solution: (a) is straight-forward. No one had trouble there. For (b), it will suffice to show that any subset $Z \subseteq X$ which is closed in the topology τ_q is closed in the topology τ. Since (X, τ) is first countable, it suffices to show that $x \in Z$ whenever there is a sequence $z_1, z_2, \dots \in Z$ converging to x in X. Notice that $K := \{x, z_1, z_2, \dots\}$ is compact because the assumed convergence means any neighborhood of x contains all but finitely many z_i, so the assumption that Z is closed in the τ_q topology implies that $Z \cap K$ is closed in K. But $Z \cap K$ certainly contains $\{z_1, z_2, \dots\}$ and $x \in K$ is certainly in the closure of $\{z_1, z_2, \dots\}$, so we must have $x \in Z \cap K$, hence $x \in Z$ as desired.

(7) **(5)** Let $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$ be continuous map of topological spaces with the same codomain. Let

$$X_1 \times_Y X_2 := \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\}.$$

Give $X_1 \times_Y X_2$ the subspace topology from the inclusion $X_1 \times_Y X_2 \subseteq X_1 \times X_2$ ($X_1 \times X_2$ is given the product topology). This ensures that the projections $\pi_i : X_1 \times_Y X_2 \to X_i$ ($i = 1, 2$) are continuous. Prove that

$$\pi_1 : X_1 \times_Y X_2 \to X_1$$

is open when f_2 is open. ("Open maps are stable under base change.")
Solution: It suffices to prove that the image of each basic open (in any basis for $X_1 \times_Y X_2$) under π_1 is open, so it will be enough to prove that $\pi_1(U_1 \times_Y U_2)$ is open in X_1 whenever U_1 is an open subset of X_1 and U_2 is an open subset of X_2. (Note that

$$U_1 \times_Y U_2 = (U_1 \times U_2) \cap (X_1 \times_Y X_2)$$

so such sets $U_1 \times_Y U_2$ form a basis for $X_1 \times_Y X_2$ by definition of the product and subspace topologies.) For this, a little "set theory exercise" yields the formula

$$\pi_1(U_1 \times_Y U_2) = U_1 \cap f_1^{-1}(f_2(U_2)),$$

which is open when f_2 is open, since f_1 is continuous.

(8) **(5)** The "sea urchin" *(denizkestanesi)* is the space Y obtained by "gluing together infinitely many copies of I at the origin." To be more precise, let $X := I \times \omega$ and let $\sim$ be the equivalence relation on X where $(x, m) \sim (y, n)$ iff $(x, m) = (y, n)$ or $x = y = 0$. Let $[x, m]$ denote the equivalence class of (x, m). Let $q : X \to X/\sim =: Y$, $q(x, n) := [x, n]$ be the quotient, with the quotient topology τ_q on Y. Let $y_0 := [0, 0]$ be the "special" point of Y.

We can alternatively topologize the set Y as follows: Declare a basic open neighborhood of y_0 to be a set of the form

$$U_\epsilon = \{[x, n] : x < \epsilon\},$$

for $\epsilon > 0$. For $x \in (0, 1]$ and $m \in \omega$, a basic open neighborhood of $[x, m]$ is a set of the form $\{[y, m] : y \in U\}$, for U a neighborhood of x in $(0, 1]$. These basic opens form a basis for a topology τ. (You don't need to prove this.)

(a) Prove that $\tau \subsetneq \tau_q$.
(b) Prove that (Y, τ) is metrizble, but (Y, τ_q) is not even first countable.

Solution: To see that $\tau \subseteq \tau_q$, we just need to check that each of the basic opens V in the τ-topology is open in the quotient topology τ_q—i.e. that $q^{-1}(V)$ is open in $I \times \omega$. Indeed, for the basic open U_ϵ, we have

$$q^{-1}(U_\epsilon) = [0, \epsilon) \times \omega$$

and for the other kind of basic open $V(U, m) = \{[y, m] : y \in U\}$, we have

$$q^{-1}(V(U, m)) = (I \times \{m\}) \cap (U \times \omega).$$

To see that (X, τ) is metrizable, you can use Urysohn's Metrization Theorem: To see that (Y, τ) is second countable, use the basic opens $U_{1/n}$ and $V(U, n)$ as U runs over a countable basis for $(0, 1]$ and n runs over the countable set ω. To see that (Y, τ) is regular, consider a closed subset $A \subseteq Y$ and a point $y = [x, n] \in Y \setminus A$. We need to show that y and A are contained in disjoint opens S, T. This follows easily from the fact that $(0, 1]$ is regular if $x > 0$. If $y = y_0$, then since A is closed some U_ϵ is disjoint from A and $S := U_{\epsilon/2}$, $T := \cup_{n\in\omega} V((\epsilon/2, 1], n)$ will do.

To see that (X, τ_q) isn't first countable at y_0, suppose $U_1, U_2, \ldots$ are open neighborhoods of y_0. For each $n \in \omega$, $(0, n) \in q^{-1}(y_0)$ and $q^{-1}(U_n)$ is an open subset of $I \times \omega$ containing $q^{-1}(y_0)$, so there will be some $\epsilon_n > 0$ so that $[0, \epsilon_n) \times \{n\} \subseteq q^{-1}(U_n)$. Let $V := \cup_{n\in\omega}[0, \epsilon_n/2) \times \{n\}$. Then V is an open subset of $I \times \omega$ which is $\sim$-saturated $(q^{-1}(q(V)) = V)$ because

$$(0, 1), (0, 2), \cdots \in V,$$

so $q(V)$ is an open neighborhood of y_0 in (Y, τ_q). There is no n for which $U_n \subseteq q(V)$ because there is no n for which $q^{-1}(U_n)$ is contained in $q^{-1}(q(V)) = V$.