AMERICAN UNIVERSITY OF BEIRUT
Mathematics Department
Math 218 - Quiz II
Fall 2006-2007

Name:..........Solution..........

ID:..

Section: 4 (@ 12:30) 5 (@ 9:30)

Time: 60 min

I- (a) **(10 points)** Determine whether the set W of all $n \times n$ matrices A such that $A^T = -A$ is a subspace of the set of all $n \times n$ matrices M_{nn} with standard matrix addition and scalar multiplication.

① Suppose $A \in W$, $A^T = -A$
and $B \in W$, $B^T = -B$

4

$$(A+B)^T = A^T + B^T = -A - B = -(A+B)$$

so $A+B \in W$

② Suppose $A \in W$, $A^T = -A$
and k is a scalar

4

$$(kA)^T = kA^T = k(-A) = -(kA) \text{ so } kA \in W$$

2

Therefore, W is closed under addition and scalar multiplication, that is W is a subspace of M_{nn}.

(b) **(2 points)** Is the set of all diagonal $n \times n$ matrices a subspace of W?

No, since if A is $n \times n$ A^T may be different than $-A$, so the set of all diagonal $n \times n$ matrices is not a subset of W.

II- Find a basis for:

(a) **(7.5 points)** The line $2x - y = 0$ in $\mathbb{R}^2$.

$2x - y = 0 \qquad x = t, \; y = 2t \qquad t \in \mathbb{R}$

Any point on this line has the form $(t, 2t) = t(1,2)$

Therefore $\{(1,2)\}$ is a basis for the line $2x - y = 0$.

(b) **(7.5 points)** The plane $2x - y = 0$ in $\mathbb{R}^3$

$z = t$

$y = s$

$x = \frac{1}{2}y = \frac{1}{2}s$

5 Any point in this plane has the form

$(x,y,z) = (\frac{1}{2}s, s, t) = s(\frac{1}{2}, 1, 0) + t(0,0,1)$

1½ Moreover, $\{(\frac{1}{2}, 1, 0), (0,0,1)\}$ is linearly independent since if $k_1(\frac{1}{2}, 1, 0) + k_2(0,0,1) = 0$

then $\begin{cases} \frac{1}{2}k_1 = 0 \\ k_1 = 0 \\ k_2 = 0 \end{cases}$ so $k_1 = k_2 = 0$.

1 Therefore, $\{(\frac{1}{2}, 1, 0), (0,0,1)\}$ is a basis for the plane $2x - y = 0$ in $\mathbb{R}^3$.

III- (10 points) Let $v_1 = (1,2,2,3)$, $v_2 = (0,1,0,0)$, $v_3 = (5,12,\mathbf{10},\mathbf{15})$, $v_4 = (-1,1,-2,-3)$. Find a subset of $\{v_1, v_2, v_3, v_4\}$ that is a basis for Span $\{v_1, v_2, v_3, v_4\}$.

$$A = \begin{bmatrix} 1 & 0 & 5 & -1 \\ 2 & 1 & 12 & 1 \\ 2 & 0 & 10 & -2 \\ 3 & 0 & 15 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Since $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for the column space of R, then $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for the column space of A.

Therefore $\{v_1, v_2\}$ is a basis for Span$\{v_1, v_2, v_3, v_4\}$.

IV- (a) (8 points) Given

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -2 \\ 3 & 0 & 18 \end{bmatrix}$$

Define $nullity(A)$ then find it.

2 nullity (A) = dim (null space of A)

$Ax = 0$ $\quad \left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 3 & 0 & 18 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -3 & 6 & 0 \end{array}\right]$

3

$$\to \begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(continue your answer here)

$x_3 = t$
$x_2 = 2t$
$x_1 = -2t - 4t = -6t$

3 $(x_1, x_2, x_3) = (-6t, 2t, t) = t(-6, 2, 1)$

Therefore $\{(-6,2,1)\}$ is a basis for the nullspace of A.

nullity $(A) = 1$

(b) **(5 points)** Define $rank(A)$ then deduce it from part (a).

2 rank$(A) =$ dim (column space of A)
$=$ dim (row space of A)

3 rank(A) + nullity$(A) = 3$ (Dimension theorem)
so rank$(A) = 3 - 1 = 2$

(c) **(5 points)** Is it true that the row vectors of A do not span $\mathbb{R}^3$? Why or why not?

Yes, since nullity $(A) \neq 0$.
By the equivalences theorem, the row vectors of A do not span $\mathbb{R}^3$.

V- **(10 points)** Let T: $\mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be a linear transformation such that

$T(1,2,0) = (3,1,-1,4)$,

$T(2,1,1) = (0,2,1,-8)$,

$T(2,0,1) = (1,0,-2,0)$.

Find the standard matrix for T.

$T(1,2,0) = b_1$

$(1,2,0) = e_1 + 2e_2$ so $T(e_1) + 2T(e_2) = b_1$

$(2,1,1) = 2e_1 + e_2 + e_3$ so $2T(e_1) + T(e_2) + T(e_3) = b_2$

$(2,0,1) = e_1 + e_3$ $\quad T(e_1) + T(e_3) = b_3$

Solve the system $\left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 2 & 1 & 1 & b_2 \\ 2 & 0 & 1 & b_3 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & -3 & 1 & -2b_1 + b_2 \\ 0 & -4 & 1 & -2b_1 + b_3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc} 1 & 2 & 0 & b_1 \\ 0 & 1 & 0 & b_2 - b_3 \\ 0 & -4 & 1 & -2b_1 + b_3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 1 & 0 & b_2 - b_3 \\ 0 & 0 & 1 & 4b_2 - 3b_3 - 2b_1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 - 2b_2 + 2b_3 \\ 0 & 1 & 0 & b_2 - b_3 \\ 0 & 0 & 1 & 4b_2 - 3b_3 - 2b_1 \end{array}\right]$$

$T(e_1) = b_1 - b_2 + b_3 = (3,1,-1,4) - 2(0,2,1,-8) + 2(1,0,-2,0)$
$= (5, -3, -7, 20)$

$T(e_2) = b_2 - b_3 = (0,2,1,-8) - (1,0,-2,0)$
$= (-1, 2, 3, -8)$

$T(e_3) = 4b_2 - 3b_3 - 2b_1 = (-9, 6, 12, -40)$

The standard matrix for $T = \begin{bmatrix} 5 & -1 & -9 \\ -3 & 2 & 6 \\ -7 & 3 & 12 \end{bmatrix}$

The previous method is the standard way for such questions but here there is a much easier way.

$$(0,1,0) = (2,1,1) - (2,0,1)$$

so $T(0,1,0) = T(2,1,1) - T(2,0,1) = (0,2,1,-8) - (1,0,-2,0)$

$$= (-1,2,3,-8)$$

$$(1,0,0) = (1,2,0) - 2(0,1,0)$$

$$T(1,0,0) = T((1,2,0) - 2(0,1,0)) = T(1,2,0) - 2T(0,1,0)$$

$$= (3,1,-1,4) - 2(-1,2,3,-8)$$

$$= (3,1,-1,4) + (2,-4,-6,16) = (5,-3,-7,20)$$

$$(0,0,1) = (2,0,1) - 2(1,0,0)$$

$$T(0,0,1) = T(2,0,1) - 2T(1,0,0)$$

$$= (1,0,-2,0) - 2(5,-3,-7,20)$$

$$= (1,0,-2,0) + (-10,6,14,-40) = (-9,6,12,-40)$$

so the standard matrix for T is

$$[T] = [T(e_1) \quad T(e_2) \quad T(e_3)]$$

$$= \begin{bmatrix} 5 & -1 & -9 \\ -3 & 2 & 6 \\ -7 & 3 & 12 \\ 20 & -8 & -40 \end{bmatrix}$$

VI- DO NOT PROVE ANY OF YOUR ANSWERS IN THIS QUESTION UNLESS YOU ARE ASKED TO DO SO.

a- **(27 points)** Write **TRUE** or **FALSE**.

3 points each

1- W is a subspace of a vector space V if and only if $ku + v \epsilon V$ for all scalars k and vectors u and v in V. T

2- If the row vectors and the column vectors of A are linearly independent then A is square. T

3- If A is an invertible $n \times n$ matrix then A and AB have the same row space. T

4- In an underdetermined system $Ax = b$ where A is $m \times n$, the column vectors are linearly independent. F

5- In an overdetermined system $Ax = b$ where A is $m \times n$, the column vectors do not span $\mathbb{R}^m$. T

6- The union of two subspaces of a vector space V is always a subspace of V. F

7- If W is a subspace of a vector space V and S is a subspace of W, then S is a subspace of V. T

8- $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x, y) = (x^2, 2x + 1)$ is a linear transformation. F

9- If A is a 4×5 matrix then the smallest value for nullity(A) is 1. T

b- **(8 points)** If A and B are two $n \times n$ matrices. Is $rank(A + B) = rank(A) + rank(B)$? If yes, prove this. If no, give a counter example.

False.

Take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ clearly $rank(A) = 3$

$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ also $rank(B) = 3$

$A + B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $rank(A+B) = 0$

This means that $rank(A+B) \neq rank(A) + rank(B)$

GOOD LUCK