AMERICAN UNIVERSITY OF BEIRUT
Mathematics Department
Math 218 - Quiz II
Spring 2006-2007

Name: Solution **ID:**...

Section: **9 (@ 9:30)** **2 (@ 11:00)** **1 (@ 12:30)**

Time: 60 min

Unjustified answers will not receive any credit

I- Give a *precise* definition of the following expressions.

(a) **(4 points)** Vector Space

1 A vector space V is a nonempty set with 2 operations: ½ addition and scalar multiplication that satisfy the following axioms: for all $u, v, w \in V$ and scalars k, m.

(1) $u+v \in V$

(2) $u+v = v+u$

(3) $u+(v+w) = (u+v)+w$

(4) there is $0_V \in V$ such that $0_V + u = u$ for all $u \in V$

(5) there is $-u \in V$ for all $u \in V$ such that $u+(-u) = 0_V$

(6) $ku \in V$

(7) $(k+m)u = ku + mu$

(8) $k(u+v) = ku + kv$

(9) $k(mu) = (km)u$

(10) $1u = u$

2½

(b) **(4 points)** Span $\{v_1, v_2, ..., v_n\}$

Span $\{v_1, \dots v_n\}$ is the set of all linear combinations 3
1 of $v_1, \dots v_n$, i.e,

$$\text{Span}\{v_1, \dots v_n\} = \{k_1 v_1 + k_2 v_2 + \dots k_n v_n \mid k_1, \dots k_n \text{ are scalars}\}$$

(c) **(4 points)** Linear Transformation

A linear transformation is a function 1 $T: \mathbb{R}^n \to \mathbb{R}^m$ that can be represented by a matrix, that is $T(x) = Ax$ for all x 3

or

It is a function $T: \mathbb{R}^n \to \mathbb{R}^m$ satisfying the following conditions:

$T(u+v) = T(u) + T(v)$ and $T(cu) = cT(u)$ for all $c \in \mathbb{R}$, $u, v \in \mathbb{R}^n$

(d) **(4 points)** Column Space of a matrix A

The column Space of a matrix A is the span of the column vectors of A.

II- Consider the vector space M_{22} with standard addition and scalar multiplication. Let W be the subspace of all matrices of the form

$$\begin{bmatrix} x & y+z \\ 3y & 2z-x \end{bmatrix}$$

such that $(x,,y,z) \in NullSpace_B$ where $B = \begin{bmatrix} 6 & 12 & -30 \\ -5 & -10 & 25 \\ 2 & 4 & -10 \end{bmatrix}$.

(a) **(9 points)** Find a basis for W.

Step 1: $(x,y,z) \in NullSpace_B \Rightarrow B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \left[\begin{array}{ccc|c} 6 & 12 & -30 & 0 \\ -5 & -10 & 25 & 0 \\ 2 & 4 & -10 & 0 \end{array}\right]$

3 $\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -5 & 0 \\ 1 & 2 & -5 & 0 \\ 1 & 2 & -5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x+2y-5z=0$

$y = t$

$z = s$

$x = -2t+5s$

3 $\begin{bmatrix} x & y+z \\ 3y & 2z-x \end{bmatrix} = \begin{bmatrix} -2t+5s & t+s \\ 3t & 2t-3s \end{bmatrix} = t\begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} + s\begin{bmatrix} 5 & 1 \\ 0 & -3 \end{bmatrix}$

2 Step 2: $k_1\begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} + k_2\begin{bmatrix} 5 & 1 \\ 0 & -3 \end{bmatrix} = 0 \Rightarrow$ $\begin{array}{l} -2k_1+5k_2=0 \\ k_1+k_2=0 \\ 3k_1=0 \\ 2k_1-3k_2=0 \end{array}$ $\Rightarrow k_1=k_2=0$

Step 3: 1

$\left\{\begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 0 & -3 \end{bmatrix}\right\}$ is a basis for W.

(b) **(2 points)** Find $dim(W)$.

$\dim(W) = 2$

(c) **(4 points)** Find a set of 4 vectors that spans W.

1 Take any linear combination of $\begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$ and $\begin{bmatrix} 5 & 1 \\ 0 & -3 \end{bmatrix}$. So, we can take

3 $\left\{\begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 3 & -1 \end{bmatrix}\right\}$

III- Part A:

(10 points) Given the vectors $v_1, v_2, ..., v_n$ in a vector space V. Show that $\text{Span}\{v_1, v_2, ..., v_n\}$ is a subspace of V.

$\text{Span}\{v_1, v_2, --- v_n\} = \{k_1 v_1 + --- k_n v_n \mid k_1, --- k_n \text{ are scalars}\}$

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① Since $v_1, -- v_n$ are in V then $k_1 v_1 + --- k_n v_n \in V$

so $\text{Span}\{v_1, -- v_n\} \subseteq V$

3 ② If $u, v \in \text{Span}\{v_1, --- v_n\}$

$u = k_1 v_1 + --- k_n v_n$ $\qquad$ $u + v = (k_1 + c_1) v_1 + -- (k_n + c_n) v_n$

$v = c_1 v_1 + --- c_n v_n$ $\qquad$ $\in \text{Span}\{v_1, -- v_n\}$

3 ③ If $u \in V$ and $k \in \mathbb{R}$.

$u = k_1 v_1 + --- k_n v_n$ $\qquad$ $ku = kk_1 v_1 + --- kk_n v_n \in \text{Span}\{v_1, - v_n\}$

Therefore, $\text{Span}\{v_1, --- v_n\}$ is a subspace of V

Part B:

3 points for the rule

Let $v_1 = (1, 3, 1, -3)$, $v_2 = (2, 7, 3, -6)$, $v_3 = (-4, -15, -7, 12)$, $v_4 = (5, 17, 7, -15)$.

(a) **(12 points)** Find a basis for the space spanned by v_1, v_2, v_3, v_4 that is a subset of $\{v_1, v_2, v_3, v_4\}$.

$$A = \begin{bmatrix} 1 & 2 & -4 & 5 \\ 3 & 7 & -15 & 17 \\ 1 & 3 & -7 & 7 \\ -3 & -6 & 12 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -4 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 1 & -3 & 2 \\ 0 & 1 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -4 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

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Therefore, $\{v_1, v_2\}$ is a basis for $\text{Span}\{v_1, v_2, v_3, v_4\}$

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(b) **(10 points)** Write each vector that is not in the basis as a linear combination of elements in the basis.

$$\begin{bmatrix} -4 \\ -3 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{v_3 = 2v_1 - 3v_2}$$

5

$$\begin{bmatrix} 5 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{v_4 = v_1 + 2v_2}$$

5

7 points for the correct answer using systems and without showing the solution.

IV- (18 points) Consider two linear transformations T_1 and T_2 where the standard matrices representing them are A_1 and A_2 respectively. Given that the standard basis of $\mathbb{R}^3$ is $\{e_1, e_2, e_3\}$, we know the following:

$$T_2(e_2) = T_1(3e_1) + (2,1,1,5)$$
$$T_2(2e_2) - 5T_1(e_1) = (5,0,3,10)$$

Find all possible one-to-one linear transformations T such that if A is the standard matrix representing T then the first column of $A - A_1$ and the second column of $A - A_2$ are all zeros.

(Note: To determine a group of linear transformations it is enough to give the general form of their standard matrix.)

Let $T_2(e_2) = x$, $T_1(e_1) = y$ $\quad b_1 = (2,1,1,5) \quad b_2 = (5,0,3,10)$

$$\begin{cases} x - 3y = b_1 \\ 2x - 5y = b_2 \end{cases} \Rightarrow x = \frac{\begin{vmatrix} b_1 & -3 \\ b_2 & -5 \end{vmatrix}}{\begin{vmatrix} 1 & -3 \\ 2 & -5 \end{vmatrix}} = -5b_1 + 3b_2$$

$$= (-10,-5,-5,-25) + (15,0,9,30)$$

and $\boxed{x = (5,-5,4,5)}$ 5

and $y = \frac{\begin{vmatrix} 1 & b_1 \\ 2 & b_2 \end{vmatrix}}{\begin{vmatrix} 1 & -3 \\ 2 & -5 \end{vmatrix}} = b_2 - 2b_1$

$$= (5,0,3,10) + (-4,-2,-2,-10)$$

$\boxed{y = (1,-2,1,0)}$ 5

The first column of A is equal to the first column of A_1 so it is $T_1(e_1)$ similarly the second column of A is $T_2(e_2)$

so $A = \begin{bmatrix} 1 & 5 & a \\ -2 & -5 & b \\ 1 & 4 & c \\ 0 & 5 & d \end{bmatrix}$ 5

For T to be one-to-one $Ax = 0$ should have only the trivial solution. 3

$$\left[\begin{array}{ccc|c} 1 & 5 & a & 0 \\ -2 & -5 & b & 0 \\ 1 & 4 & c & 0 \\ 0 & 5 & d & 0 \end{array}\right] \to \left[\begin{array}{cccc} 1 & 5 & a & 0 \\ 0 & 5 & 2a+b & 0 \\ 0 & -1 & c-a & 0 \\ 0 & 5 & d & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 5 & a & 0 \\ 0 & 1 & a-c & 0 \\ 0 & 5 & d & 0 \\ 0 & 0 & 2a+b-d & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 5 & a & 0 \\ 0 & 1 & a-c & 0 \\ 0 & 0 & -5a+5c+d & 0 \\ 0 & 0 & 2a+b-d & 0 \end{array}\right]$$

So we need $-5a+5c+d$ and $2a+b-d$ not both zero.

V- Answer the following questions

(a) **(8 points)** If the row vectors of a matrix A are linearly independent, and the column vectors are also linearly independent, explain why A is a square matrix.

Say A is $m \times n$ $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$

There are n column vectors in $\mathbb{R}^m$ that are linearly independent so $\boxed{n \leq m}$

Also, there are m row vectors in $\mathbb{R}^n$ that are linearly independent so $\boxed{m \leq n}$

Hence, $\boxed{m = n}$

(b) **(10 points)** W is a space spanned by the elements $\{v_1, v_2, ..., v_n\}$ of a vector space V. We know that if we want to check the linear independence of $\{v_1, v_2, ..., v_n\}$ we have to solve a system of 4 equations. Moreover, if we put $v_1, v_2, ..., v_n$ as rows of a matrix A, then the row echelon form of A has exactly 4 non-zero rows. Determine W precisely. Justify your conclusion.

To check linear independence we need to put $k_1 v_1 + k_2 v_2 + \cdots k_n v_n = 0$

So if we have 4 equations then

3.5 $v_1, \cdots v_n$ have 4 components.

Notice that $v_1 \cdots v_n$ are vectors in a Euclidean space since we could put them as rows of a matrix A. Since REF

3.5 of A has exactly 4 nonzero rows, so it has exactly 4 leading 1's.

So $v_1 \cdots v_n$ are in $\mathbb{R}^4$

GOOD LUCK

Hence, $\text{Span}\{v_1 \cdots v_n\}$ is a subspace of $\mathbb{R}^4$,

3 and its dimension is 4

Therefore, $\text{Span}\{v_1 \cdots v_n\} = \mathbb{R}^4$.