**Fall 2008-2009** **Final Exam** **Date: January 29**
**Prof: M. Egeileh** **Math 218 - Section 7** **Duration: 2 hours**

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**Exercise 1** The vector space $M_2(\mathbb{R})$ being equipped with its standard inner product (defined by $<A,B>= \text{tr}(({}^tA)B)$ for any $A, B \in M_2(\mathbb{R})$), we consider the following two matrices : $A = \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3\sqrt{3} & 0 \\ 0 & 3 \end{pmatrix}$.

1. Compute $\|A\|$ and $\|B\|$. *(10 pts)*
2. Find the angle between the matrices $A$ and $B$. *(10 pts)*

**Exercise 2** In this exercise, $\mathcal{B} = \{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$ and $\mathcal{B}' = \{e'_1, e'_2\}$ is the standard basis of $\mathbb{R}^2$. We consider the linear map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by :
$T(x_1, x_2, x_3) = (x_1 - x_2 + 3x_3\ ,\ 4x_1 - 4x_2 + 12x_3)$ for every $(x_1, x_2, x_3) \in \mathbb{R}^3$.

1. Let $A = [T]_{\mathcal{B}',\mathcal{B}}$ be the representative matrix of $T$ in the bases $\mathcal{B}$ and $\mathcal{B}'$.
   Write $A$ (find all its entries). *(5 pts)*
2. Check that $T(x) = Ax$ for every $x \in \mathbb{R}^3$ (here, $x$ and $T(x)$ are written as column vectors). *(3 pts)*
3. Find a basis for the kernel of $T$. *(6 pts)*
4. What is the rank of $T$ ? Justify your answer. *(4 pts)*
5. Find a basis for the image of $T$. *(6 pts)*
6. Is $T$ injective ? surjective ? bijective ? Justify your answers. *(6 pts)*

**Exercise 3** Let $V$ be a 3-dimensional vector space, and $\mathcal{B} = \{a_1, a_2, a_3\}$ a basis of $V$. We consider the linear operator $T : V \longrightarrow V$ whose representative matrix in $\mathcal{B}$ is the following matrix : $A = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. The two parts of the problem can be treated independently ; however, it is recommended to solve Part I before Part II.

**Part I :**

1. Find the eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$ of $T$. *(6 pts)*
2. Without performing any computation, find, for each eigenvalue $\lambda_i$ you obtained in 1., the dimension of the corresponding eigenspace $\text{Ker}(T - \lambda_i \text{Id}_V)$ (*Hint : remember that for any eigenvalue $\lambda$, dim Ker$(T - \lambda Id_V)$ (the geometric multiplicity of $\lambda$) is* **not** *greater than the multiplicity of $\lambda$ as root of the characteristic polynomial of $A$*). *(5 pts)*
3. Deduce that the linear operator $T$ is diagonalizable. *(4 pts)*
4. Find, for each eigenvalue $\lambda_i$, a basis of the corresponding eigenspace $\text{Ker}(T - \lambda_i \text{Id}_V)$. *(6 pts)*

**Part II :**

1. Let $a'_1, a'_2, a'_3$ be the vectors of $V$ defined by : $a'_1 = a_1$, $a'_2 = a_1 + a_2$, $a'_3 = a_1 + a_3$.
   Check that $\mathcal{B}' = \{a'_1, a'_2, a'_3\}$ is a basis of $V$. *(2 pts)*
2. By expressing the vectors $T(a'_1)$, $T(a'_2)$, $T(a'_3)$ in $\mathcal{B}'$, write the matrix $D = [T]_{\mathcal{B}'}$. *(2 pts)*
3. Write the matrices $[\text{Id}_V]_{\mathcal{B},\mathcal{B}'}$ and $[\text{Id}_V]_{\mathcal{B}',\mathcal{B}}$. *(2 pts)*

4. What matrix would you obtain if you compute the product $[\mathrm{Id}_V]_{\mathcal{B},\mathcal{B}'} \cdot D \cdot [\mathrm{Id}_V]_{\mathcal{B}',\mathcal{B}}$? Justify your answer (do **not** evaluate the product array-wise in **this** question). *(2 pts)*
5. Check the result of the preceding question by computing explicitly the product of matrices $[\mathrm{Id}_V]_{\mathcal{B},\mathcal{B}'} \cdot D \cdot [\mathrm{Id}_V]_{\mathcal{B}',\mathcal{B}}$. *(2 pts)*
6. Give a basis of $\mathrm{Im}\, T$ made of vectors expressed in $\mathcal{B}'$. *(2 pts)*
7. Give a basis of $\mathrm{Im}\, T$ made of vectors expressed in $\mathcal{B}$. *(2 pts)*

**Exercise 4** Let $W$ be the vector subspace of $\mathbb{R}^3$ spanned by $\mathcal{S} = \{b_1, b_2\}$, where $b_1 = (1, 0, 0)$ and $b_2 = (-2, 0, -1)$.

1. Check that $\mathcal{S}$ is a basis of $W$. *(2 pts)*
2. $\mathbb{R}^3$ is equipped with its standard Euclidian structure. Check that $\mathcal{S}$ is **not** an orthonormal basis of $W$. *(2 pts)*
3. Let $L$ be the vector line spanned by $b_1$, and $p_L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ the orthogonal projection on $L$. Find a formula giving $p_L(x)$ for any $x \in \mathbb{R}^3$. *(3 pts)*
4. Apply Gram-Schmidt process on $\mathcal{S}$ to construct an orthonormal basis $\{c_1, c_2\}$ for the vector subspace $W$. *(5 pts)*

**Exercise 5** In the vector space $\mathcal{P}_9(\mathbb{R})$ of polynomials whose degree is less or equal to 9, we consider the vector subspace $W = \mathcal{P}_4(\mathbb{R})$. Find the rank and the trace of the linear operator $T_W : \mathcal{P}_9(\mathbb{R}) \longrightarrow \mathcal{P}_9(\mathbb{R})$ which associates to every polynomial $p$ defined by $p(x) = a_0 + a_1 x + \ldots + a_9 x^9$ the polynomial $T_W(p)$ defined by $T_W(p)(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$. *(3 pts)*