AMERICAN UNIVERSITY OF BEIRUT
Mathematics Department
Math 218 - Final Exam
Fall 2006-2007

Name:.......................................

ID:..

Section: 4 (@ 12:30) 5 (@ 9:30)

Time: 120 min

Directions:

- Read each question carefully before you answer.
- If you are asked to answer by TRUE or FALSE, make sure that you do not give an abrupt answer.

I- Give a ***precise*** definition of the following:

(a) **(4 points)** A vector space:

(b) **(3 points)** A basis for a vector space:

(c) **(3 points)** The rank of a matrix:

(d) **(3 points)** A linear transformation between two vector spaces:

(e) **(3 points)** The nullity of a linear transformation:

II- Let

$$A = \begin{bmatrix} 1 & 4 & -3 & 0 \\ 2 & 10 & 0 & 4 \\ 3 & 13 & -7 & 3 \end{bmatrix}$$

(a) **(2.5 points)** Find a basis for the row space of A.

(continue your answer here)

(b) **(2.5 points)** Find a basis for the column space of A.

(c) **(3 points)** Use (b) to show that the system

$$\begin{bmatrix} 1 & 4 & -3 \\ 2 & 10 & 0 \\ 3 & 13 & -7 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$$

is consistent, ***without solving it***.

III- Let

$$\mathrm{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) **(4 points)** Find the eigenvalues of B.

(c) **(8 points)** Find bases for the eigenspaces of B.

(Continue your answer here)

(b) **(5 points)** Is B diagonalizable? If yes, find a matrix P that diagonalizes B and determine $P^{-1}BP$.

IV- Consider the system $Ax = b$.

(a) (i) **(6 points)** Find a least squares solution for the system if:

$$\text{A} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \text{ and b} = \begin{bmatrix} 6 \\ 3 \\ 9 \\ 6 \end{bmatrix}$$

(Continue your answer here)

(ii) **(4 points)** Find the orthogonal projection of $u = (6, 3, 9, 6)$ on the subspace of $\mathbb{R}^4$ spanned by the vectors $v_1 = (2, 1, 1, 1)$, $v_2 = (1, 0, 1, 1)$, and $v_3 = (-2, -1, 0, -1)$.

(b) **(3 points)** If b is in the null space of A^T. What is $Proj_W b$? Justify.

V- Let T be a function from the vector space of all $n \times n$ matrices M_{nn} to itself with standard addition and scalar multiplication. Let C be a fixed $n \times n$ invertible matrix. Define T by $T(A) = CA$ for all $A \in M_{nn}$.

(a) **(4 points)** Show that T is a linear transformation.

(b) **(5 points)** Show that T is one-to-one.

(c) **(5 points)** Deduce $rank(T)$.

(Continue your answer here)

(c) **(3 points)** Is T is an isomorphism? Explain.

VI- Let T be a linear transformation from the vector space P_2 to the vector space $\mathbb{R}^2$ with standard addition and scalar multiplication defined by:

$$T(p(x)) = (\, p(0)\, ,\, p(-1) + p(1)\,)$$

(a) **(5 points)** Let $B = \{1, x^2, x\}$ and $B' = \{e_1, e_2\}$. Find ${}_{B'}[T]_B$.

(b) **(5 points)** Define, then find $ker(T)$.

VII- **(8 points)** If $T : V \longrightarrow W$ is a one-to-one linear transformation and $\{v_1, v_2, ..., v_n\}$ is a basis for V, then $\{T(v_1), T(v_2), ..., T(v_n)\}$ is linearly independent in W.

VIII- (5 points) Specify the appropriate vector space, operations and inner product , then, use Cauchy Schwarz inequality to show that for all positive numbers $a_1, a_2, ..., a_n$,

$$(a_1 + 2a_2 + ... + na_n)\left(\frac{1}{a_1} + \frac{2}{a_2} + ... + \frac{n}{a_n}\right) \geqslant \frac{n^2(n+1)^2}{4}$$

IX- (6 points) Write **TRUE** or **FALSE**.

1- If λ is an eigenvalue of an invertible matrix A, then $\dfrac{1}{\lambda}$ is an eigenvalue of A^{-1}.

2- If v is an eigenvector of a square matrix A, then it is also an eigenvector of A^5.

3- If $det(A) = \sqrt{3}$ then $\lambda = 0$ is an eigenvalue of A.

4- If $T : \mathbb{R}^5 \longrightarrow M_{33}$ and $rank(T) = 5$ then T is not one-to-one.

5- The matrix $A^T A - 3I$ is orthogonally diagonalizable for any matrix A.

6- If $\{v_1, v_2, ..., v_n\}$ is an orthogonal set of nonzero vectors in an inner product space V such that $dim(V) = m$, then $m \leqslant n$.

GOOD LUCK

If you have any comments or requests, please write them here: