AUB - Math 218 - Spring 2008 FINAL EXAM, June 4 Duration: 2 hours

Instructor: N. Makhoul-Karam Section 1

Exercise 1: (20 points: 5 for each question)

a) Suppose that $p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3$ is the characteristic polynomial of some matrix *A*.

- What is the size of *A*? *Justify your answer.*
- Is *A* invertible? *Justify your answer.*

b) Find the rank of the matrix $B = \begin{pmatrix} -2 & 4 & 1 \\ 5 & 0 & 1 \\ 9 & -2 & 11 \end{pmatrix}$

c) Is the function $T : R^2 \to R^2$ that swaps vector components: $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$ a linear transformation?

d) Consider the matrix $F = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

- Show that *F* is not orthogonal.
- Find a scalar *k* such that *kF* is orthogonal. *Explain your reasoning.*

Exercise 2: (10 points)
Consider the vector space $\mathbf{P_2}$, consisting of all polynomials of degree ≤ 2 (with the standard addition and scalar multiplication of functions) and let $p_1 = 2t^2 + 4t - 1$, $p_2 = t^2 - 4t + 2$ and $p_3 = t^2 + 3t + 6$.
Write the polynomial $p = t^2 + 2t + 3$ as a linear combination of p_1, p_2, p_3.

Exercise 3: (10 points) Let T: $\mathbf{R}^3 \to \mathbf{R}^3$ be multiplication by $A = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ -2 & 2 & 0 \end{pmatrix}$.

Show that the kernel of T is a line through the origin and find parametric equations for it.

Exercise 4: (15 points)
Let $u_1 = (1,2,-1)$, $u_2 = (0,1,-1)$ and $u_3 = (3,-7,1)$. Show that the set $\{u_1, u_2, u_3\}$ forms a basis for $\mathbf{R}^3$.
Then use the Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into an orthonormal basis.

Exercise 5: (20 points) Consider the matrix $A = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$.

a) Justify that *A* is orthogonnaly diagonalizable. (2 points)
b) Find the eigenvalues of *A* and an eigenvector corresponding to each eigenvalue. (12 points)
c) After normalizing those eigenvectors, deduce the orthogonal matrix P that diagonalizes *A* and give the diagonal matrix $D = P^{-1}AP$. (8 points)

Exercise 6: (25 points: 5 for each question) Briefly prove the following statements:

a) If W is a subspace of an inner product space V, then the only common vector to *W* and $W^{\perp}$ is **0**.

b) The matrix $U = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 3 \end{pmatrix}$ is diagonalizable.

c) For a given $n \times n$ matrix *A*: *A* orthogonal $\Rightarrow$ $\det(A) = \pm 1$.

d) If $p(\lambda)$ is the characteristic polynomial of an $n \times n$ matrix A, then: $\det(A) = (-1)^n p(0)$.

e) If λ is an eigenvalue of *A*, ***x*** is a corresponding eigenvector, and k is a scalar, then $(\lambda - k)$ is an eigenvalue of $(A - kI)$ and ***x*** is a corresponding eigenvector.