AUB - <u>Math 218</u> - Fall 2008-2009 <u>FINAL EXAM</u>, January 29 Duration: 2 hours

Instructor: N. Makhoul-Karam Section 9

<u>Exercise 1</u>: 12 points (6 points + 6 points)

a) Verify that the vectors $v_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$, $v_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ and $v_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ form an orthonormal basis for R^3 with the Euclidean inner product.

b) Express the vector $u = (1,1,1)$ as linear combination of v_1, v_2 and v_3.

<u>Exercise 2</u>: 12 points (3 points + 9 points) Let $A = \begin{pmatrix} 1 & 0 \\ 3 & -1 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

a) Is the linear system $Ax = b$ consistent?

b) Find the orthogonal projection of b onto the column space of A.

<u>Exercise 3</u>: 12 points (8 points + 4 points)

Consider the matrix $A = \begin{pmatrix} 0.6 & 0.8 \\ 0.4 & 0.2 \end{pmatrix}$.

a) Find a matrix P that diagonalizes A, and determine $P^{-1}AP$.

b) Prove that A^n tends to a limiting matrix $A_L = \begin{pmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix}$ as $n \to \infty$.

<u>Exercise 4</u>: 12 points Find a 3×3 matrix A that has eigenvalues $\lambda = 0, 1$ and -1 with corresponding eigenvectors $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ respectively.

<u>Exercise 5</u>: 12 points (3 points + 9 points)

Consider the polynomials $p_1 = 1 + x$ and $p_2 = 1 - x$.

a) Show that the set of polynomials $\{p_1, p_2\}$ form a basis for $\mathbf{P_1}$.

b) Let L be the linear transformation from $\mathbf{P_1}$ to $\mathbf{M_{22}}$ such that

$$L(1+x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } L(1-x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad \text{Find } L(3+x).$$

<u>Exercise 6</u>: 16 points (3 points + 13 points) Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix}$.

a) Justify that the matrix A has a QR-decomposition.

b) Find the matrices Q and R of this decomposition.

<u>Exercise 7</u>: 24 points (4 points for each question)

a) Find two distinct non-zero vectors u and v in $\mathbf{R}^2$, such that $< u, v >= 0$ for every weighted inner product on $\mathbf{R}^2$.

b) Find two distinct non-zero vectors u and v in $\mathbf{R}^2$, such that $< u, v > \neq 0$ for any inner product on $\mathbf{R}^2$.

c) Suppose that $\{v_1, \dots, v_n\}$ is an orthonormal basis of a real inner product space V. Show that for every $u \in V$, we have $\|u\|^2 = < u, v_1 >^2 + \cdots + < u, v_n >^2$.

d) Let u and v be vectors in an inner product space.
Prove that $\|u\| = \|v\|$ if and only if $u + v$ and $u - v$ are orthogonal.

e) Let A be a square matrix such that $A^3 = A$. What can you say about the eigenvalues of A?

f) Show that the matrix $P = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \end{pmatrix}$ is orthogonal.

<u>Definition</u>: A proper orthogonal matrix is an orthogonal matrix of which the determinant is equal to 1.
Check that P is not proper and deduce from P a proper orthogonal matrix P'. (Use a relevant small change only)