Lebanese American University
School **of Engineering and Architecture**
Department of Electrical and **Computer Engineering**

COE 788, Information and Coding Theory
Exam 1, Spring 2007
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Problem 1: (**15 points**: (6×2.5))
Consider the two discrete random variables X and Y defined over a given finite alphabet composed of M distinct elements. Indicate when will the following equalities hold:

1. $H(X) = 0$.
2. $H(X,Y) = H(X) + H(Y)$.
3. $I(X,Y) = 0$.
4. $H(Y) = H(X)$.
5. $H(Y|X) = H(Y)$.
6. $H(X) = \log_2(M)$ bits.

Remark: For this question, you must only provide the answer. No partial credit will be given.

Problem 2: (**35 points**: $(10, 3, 3, 4, 7, 5, 3)$)
Consider the two discrete random variables X and Y defined over a given finite alphabet.

1. Prove the following relation:

$$H(X,Y) = H(X) + H(Y|X)$$

   Note that in this part, you must provide a detailed proof. You must first write down the expression of $H(X,Y)$ and then simplify in order to obtain the previous result...

2. Interpret the above equality.
3. Can we write $H(X,Y) = H(Y) + H(X|Y)$? Justify your answer.

4. Prove that $H(Y|X) \geq H(Y) - H(X)$.
5. In this part, $Y = f(X)$. Show that the entropy of a function of X is less than the entropy of X by justifying the following steps:

$$\begin{aligned}
H(X, f(X)) &= H(X) + H(f(X)|X) &(1)\\
&= H(X) &(2)\\
H(X, f(X)) &= H(f(X)) + H(X|f(X)) &(3)\\
&\geq H(f(X)) &(4)
\end{aligned}$$

   Then $H(f(X)) \leq H(X)$.
6. Sketch Venn diagram of X and Y when $Y = f(X)$. What is the value of $I(X,Y)$? Interpret this result.
7. When will the inequality $H(f(X)) \leq H(X)$ become an equality? Comment.

Problem 3: (**25 points**: $(2.5, 2.5, 5, 10, 5)$)
Consider the three discrete random variables X, Y and Z defined over a given finite alphabet $\mathcal{H}$.

1. Consider any two probability distributions $p(x)$ and $q(x)$ defined over $\mathcal{H}$. Indicate when will the informational divergence between these distributions be equal to zero:

$$D(p,q) = 0$$

2. From part 1, deduce when will $D(p(x,y), p(x)\,p(y)) = 0$.
3. Indicate when will the following relation hold:

$$D(p(x,y,z), p(x)\,p(y)\,p(z)) = 0$$

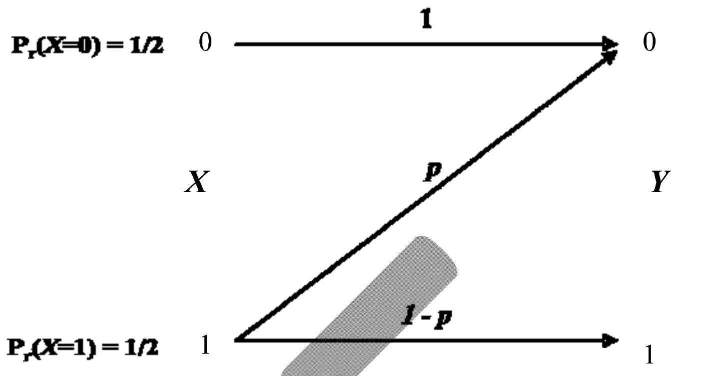
   Justify your answer.
4. Prove the following equality:

$$D(p(x,y,z), p(x)\,p(y)\,p(z)) = H(X) + H(Y) + H(Z) - H(X,Y,Z)$$

   Note that in this part, a detailed proof must be provided.
5. From part 4, deduce when will $D(p(x,y,z), p(x)\,p(y)\,p(z))$ be equal to zero. Is this result coherent with the result obtained in part 3?

Problem 4: (**25 points**: $(4, 3, 7, 6, 5)$)
The so-called Z-channel is often used as a model for optical communications. This is a binary input binary output channel where the bit "0" is always detected without error. Erroneous decisions can occur only on the bit "1" with a

probability p. We denote by X and Y the input and the output of the channel respectively.



In what follows, we assume that the probability distribution of the input is given by: $\mathrm{P}_r(X=0) = \mathrm{P}_r(X=1) = \frac{1}{2}$.

1. a. Determine, as a function of p, the values of $\mathrm{P}_r(Y=0)$ and $\mathrm{P}_r(Y=1)$. From this probability distribution of Y deduce that the entropy of Y is equal to:

$$H(Y) = H_2\left(\frac{1+p}{2}\right)$$

   where $H_2(p) = p\log\frac{1}{p} + (1-p)\log\frac{1}{1-p}$.

   b. Can we write $H(Y)$ as $H(Y) = H_2\left(\frac{1-p}{2}\right)$? Justify your answer.

2. Prove that the conditional entropy $H(Y|X)$ can be written as:

$$H(Y|X) = \sum_x p(x) H(Y|X=x)$$

   where:

$$H(Y|X=x) = \sum_y p(y|x)\log\frac{1}{p(y|x)}$$

   Note that in this part a detailed proof must be provided.

3. a. Show that:

$$H(Y|X=0) = 0$$

   Comment this result.

   b. Show that:

$$H(Y|X=1) = H_2(p)$$

   Comment this result.

c. From parts 2, 3.a and 3.b deduce that:

$$H(Y|X) = \frac{1}{2}H_2(p)$$

4. a. Show that the average amount of information that can be transmitted over the Z-channel is given by:

$$I(X,Y) = H_2\left(\frac{1+p}{2}\right) - \frac{1}{2}H_2(p)$$

b. What is the value of $I(X,Y)$ (in bits) when $p = 1$? Comment this result.

c. What is the value of $I(X,Y)$ (in bits) when $p = 0$? Comment this result. Can we say in this case that $I(X,Y) = H(X)$? Justify your answer.

5. In this part, we fix $p = \frac{1}{2}$. Calculate (in bits) the values of $H(Y)$, $H(Y|X = 0)$, $H(Y|X = 1)$ and $H(Y|X)$. Compare the following pairs of information measures:

1) $H(Y|X = 0)$ and $H(Y)$.
2) $H(Y|X = 1)$ and $H(Y)$.
3) $H(Y|X)$ and $H(Y)$.

Does knowledge always reduce uncertainty or knowledge reduces uncertainty in average. Comment.

Problem 5: (**8 points**) *Extra credit question*
Generalize the equality given in problem 3 (part 4) to any number n of discrete random variables $X_1, \ldots, X_n$. Deduce an upper bound on the joint entropy of n random variables and discuss when will this bound be achieved.