**Rank of a matrix, Linear equations, and Vectors in** $R^{n}$

*Definition:* Let A be an $\left[5×4\right]$ $\left[m×n\right] $matrix. Let RRE(A) be the reduced-row-echelon form of A.

We define the **rank** of A to be the number of non-zero rows in RRE(A) (or any echelon form of A).

Or equiv., **rank**(A) is the number of leading 1’s in RRE(A) (or the number of pivots in any echelon form of A).

* Example 1 : Let . Then **rank**(A)= 2

**vv/imp Remark**: $ rank\left[m×n\right]\leq m \& rank\left[m×n\right] \leq n $(Why? Number of leading 1’s is at most n).

* Example 2: True-False: It is impossible for a 5x4 matrix to have rank 5! True: rank $\left[5×4\right]\leq 4$.

**Theorem 1:** A linear system AX=B is consistent for all B in$ R^{m}$ iff (if and only if)

 rankA= m (the number of rows of A).

 *Idea of Proof.*  If rank (A) < m, then RRE(A) has a row of zeroes . So the last row of RRE(A|B) may look like (0 0 0 …|1 ) & we can not solve the system in such a case. If rank (A)=m, then we have no problem.

**Theorem 2:** A linear system AX=0 implies X=0 (only ), or X must be 0, iff

rankA= n (the number of columns of A).

 *Idea of Proof.*  If rank (A) < n, then the number of leading 1’s in RRE(A) is less than the number of columns of A.. So some column $C\_{m}$ of RRE(A) has no leading 1.

Hence $x\_{m}$ is an arbitrary parameter. So X is not necessarily 0.

* Example/**Problem 3**: True-False. Let A=$\left[5×4\right]$. Then it is possible for the linear system

 AX=B to be consistent (has a solution) for all B in$ R^{5}$

False Justification: AX=B is consistent for all B in$ R^{5}$ iff rank (A) = number of rows of A=5

 (by Theorem 1). But rank $\left[5×4\right]\leq 4$ . So it is impossible.

* Example/**Problem 4**: True-False. Let A=$\left[5×6\right]$. Then it is possible that the linear system

AX=0 implies X=0 (only ), i.e X must be 0,

False: Justification: AX=0 implies X=0 (only ), or X must be 0, iff rank (A) = number of columns of A=6 (by Theorem 2). But rank $\left[5×6\right]\leq 5$. So it is impossible.

**vv/imp Remark** : The linear system AX=B is consistent for a particular B in$ R^{m}$ iff

rank A=rank (A|B)

*Idea of*  *proof.* For otherwise, RRE(A|B) may look like , so the system inconsistent.

* Example/**Problem 5**: True False : In a linear system, it is possible that rank(A|B)=rank A+2.

False Justification: rank RRE (A|B) = either rank A OR rank A + 1 because

RRE(A|B) has only one more column than RRE(A). So the number of leading 1’s may go up by at most 1.

**Extra Notes/Problems on Sections 2.2 & 2.3**

**LINK**: $ x\_{1}C\_{1}+ x\_{2}C\_{2}+…..+x\_{n}C\_{n}= \left(C\_{1} \right|C\_{2}\left|….. \right|C\_{n}) X where X=the column vector of x\_{1},…,x\_{n} $

 Theorem 1 & Theorem 2 & Link have the following great consequences for vectors in $ R^{n}$.

**Golden Rule 1:** Fewer than n elements in $ R^{n}$ (automatically) can never generate$ R^{n}$.

**Golden Rule 2:** More than n elements in $ R^{n}$ are (automatically) linearly dependent.

 *Automatically means : guaranteed: you do not need to look at the vectors.*

* **Problem 6**: True –False

a) It is possible for 4 vectors in $ R^{5}$ to generate $ R^{5}$ !

b) Any 5 vectors in $ R^{4}$ are linearly dependent !

c) It is impossible for 3 vectors in $ R^{4}$ to generate $ R^{4}$ !

d) Any 3 (2x2) matrices can never generate $M\_{2×2}$ (all 2x2 matrices)!
e) Any 5 (2x2) matrices are linearly dependent in $M\_{2×2}$!

f) Any 3 polynomials in $P\_{3}$ can never generate $P\_{3}$ (all polynomials with degree at most 3)!

* **Solved Problem 7**: True –False (*What if the number of vectors is correct : no more & no less)?*

a) Any n vectors in $ R^{n}$ are linearly independent $ R^{n}$ ! Counter Example: Take i, 2i, 3i within $ R^{3}$

b) Any n vectors in $ R^{n}$ generate $ R^{n}$ ! Counter Example: Take i, 2i, 3i within $ R^{3}$

c) For n vectors in $ R^{n}$: linearly independent $⇒$ generate $ R^{n}$ ! True : This is **Golden Rule 3 (a)**

d) For n vectors in $ R^{n}$: generate $ R^{n}$ $⇒$ linearly independent! ! True : This is **Golden Rule 3 (b)**

* **Solved Problem 8:**  Prove that is impossible for 4 vectors in $ R^{5} $to generate all of $ R^{5}.$

**Yes**, this is Golden Rule 1 ( for 4 vectors in$ R^{5}$).

Proof: Let $C\_{1}, C\_{2}, …, C\_{4} be any given vectors in R^{5}. $Then the equation

$ x\_{1}C\_{1}+ x\_{2}C\_{2}+x\_{3}C\_{3}+x\_{4}C\_{4}=B in R^{5} can be written as$ $\left(C\_{1} \right|C\_{2}\left| C\_{3} \right|C\_{4})$ **X** = Bwhere X is ……

Thanks to such **LINK** ☺. Now the coefficient matrix A is$ \left[5×4\right]. $

$To solve AX=B for all B in R^{5},$ we need rank A=5 (by Theorem 1).

 But rank A= $rank \left[5×4\right]\leq 4 $. Contradiction.

* **Problem 9:**  Prove any 5 vectors in $ R^{4}$ are automatically linearly dependent.

**Yes,** this isGolden Rule 2 (for 5 vectors in$ R^{4}$).

Hint: Very similar to solved Problem 8. You need Theorem 2, LINK, and $rank \left[4×5\right]\leq 4$.

* **Problem 10:**  Prove AX=B is consistent iff B is a linear combination of the columns. (Use LINK)

Coming soon: Section 3.0 : More Golden Rules on $ R^{n}$ : Every **basis** of $ R^{n}$ has exactly n elts,…..