**Rank of a matrix, Linear equations, and Vectors in**

*Definition:* Let A be an matrix. Let RRE(A) be the reduced-row-echelon form of A.

We define the **rank** of A to be the number of non-zero rows in RRE(A) (or any echelon form of A).

Or equiv., **rank**(A) is the number of leading 1’s in RRE(A) (or the number of pivots in any echelon form of A).

* Example 1 : Let . Then **rank**(A)= 2

**vv/imp Remark**: (Why? Number of leading 1’s is at most n).

* Example 2: True-False: It is impossible for a 5x4 matrix to have rank 5! True: rank .

**Theorem 1:** A linear system AX=B is consistent for all B in iff (if and only if)

rankA= m (the number of rows of A).

*Idea of Proof.*  If rank (A) < m, then RRE(A) has a row of zeroes . So the last row of RRE(A|B) may look like (0 0 0 …|1 ) & we can not solve the system in such a case. If rank (A)=m, then we have no problem.

**Theorem 2:** A linear system AX=0 implies X=0 (only ), or X must be 0, iff

rankA= n (the number of columns of A).

*Idea of Proof.*  If rank (A) < n, then the number of leading 1’s in RRE(A) is less than the number of columns of A.. So some column of RRE(A) has no leading 1.

Hence is an arbitrary parameter. So X is not necessarily 0.

* Example/**Problem 3**: True-False. Let A=. Then it is possible for the linear system

AX=B to be consistent (has a solution) for all B in

False Justification: AX=B is consistent for all B in iff rank (A) = number of rows of A=5

(by Theorem 1). But rank . So it is impossible.

* Example/**Problem 4**: True-False. Let A=. Then it is possible that the linear system

AX=0 implies X=0 (only ), i.e X must be 0,

False: Justification: AX=0 implies X=0 (only ), or X must be 0, iff rank (A) = number of columns of A=6 (by Theorem 2). But rank . So it is impossible.

**vv/imp Remark** : The linear system AX=B is consistent for a particular B in iff

rank A=rank (A|B)

*Idea of*  *proof.* For otherwise, RRE(A|B) may look like , so the system inconsistent.

* Example/**Problem 5**: True False : In a linear system, it is possible that rank(A|B)=rank A+2.

False Justification: rank RRE (A|B) = either rank A OR rank A + 1 because

RRE(A|B) has only one more column than RRE(A). So the number of leading 1’s may go up by at most 1.

**Extra Notes/Problems on Sections 2.2 & 2.3**

**LINK**:

Theorem 1 & Theorem 2 & Link have the following great consequences for vectors in .

**Golden Rule 1:** Fewer than n elements in (automatically) can never generate.

**Golden Rule 2:** More than n elements in are (automatically) linearly dependent.

*Automatically means : guaranteed: you do not need to look at the vectors.*

* **Problem 6**: True –False

a) It is possible for 4 vectors in to generate !

b) Any 5 vectors in are linearly dependent !

c) It is impossible for 3 vectors in to generate !

d) Any 3 (2x2) matrices can never generate (all 2x2 matrices)!  
e) Any 5 (2x2) matrices are linearly dependent in !

f) Any 3 polynomials in can never generate (all polynomials with degree at most 3)!

* **Solved Problem 7**: True –False (*What if the number of vectors is correct : no more & no less)?*

a) Any n vectors in are linearly independent ! Counter Example: Take i, 2i, 3i within

b) Any n vectors in generate ! Counter Example: Take i, 2i, 3i within

c) For n vectors in : linearly independent generate ! True : This is **Golden Rule 3 (a)**

d) For n vectors in : generate linearly independent! ! True : This is **Golden Rule 3 (b)**

* **Solved Problem 8:**  Prove that is impossible for 4 vectors in to generate all of

**Yes**, this is Golden Rule 1 ( for 4 vectors in).

Proof: Let Then the equation

**X** = Bwhere X is ……

Thanks to such **LINK** ☺. Now the coefficient matrix A is

we need rank A=5 (by Theorem 1).

But rank A= . Contradiction.

* **Problem 9:**  Prove any 5 vectors in are automatically linearly dependent.

**Yes,** this isGolden Rule 2 (for 5 vectors in).

Hint: Very similar to solved Problem 8. You need Theorem 2, LINK, and .

* **Problem 10:**  Prove AX=B is consistent iff B is a linear combination of the columns. (Use LINK)

Coming soon: Section 3.0 : More Golden Rules on : Every **basis** of has exactly n elts,…..