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Mathematics 218 Final Exam.
Time: 2hrs.

1. (24 %) Prove (concisely) or Disprove (by a counter example) in an inner product space.

(a) $\|u+v\|^2+\|u-v\|^2=2\|u\|^2+2\|v\|^2$

(b) If $\|u\|\|v\|=<u,v>$ & u,v are *non-zero* then u & v are ***not*** orthogonal

(c) Non-zero orthogonal vectors are linearly independent.

(d) If two $m\times n$ matrices have the same row space, then they have the same rank and nullity.

(e) If two $m\times n$ matrices have the same row space, then they have the same null space.

(f) If A is an orthogonal $n\times n$ matrix then $\det A=\pm 1$

(g) (For any square matrix A), A and A^2 have the same row space.

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2. (20 %) (a) Find the least squares solutions of the system $\{x+y=0$ & $x+y=1$ & $x+y=4\}$

(b) What do we mean precisely by a least square (*best possible*) solution of a **non-consistent** system $AX=b$?

(c) What do we know about arbitrary symmetric $n\times n$ matrices regarding eigenvalues & diagonalization ?

(d) Apply the Cauchy-Schwarz inequality on the continous functions on the interval [0, 3].

(e) Write the orthogonal projection formula for a vector a on a subspace W (given an o.n basis of W).

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3. (15%) Let $A=\begin{bmatrix}3&6&0\\1&2&0\\0&0&5\end{bmatrix}$

(i) Find the eigen values of A and a basis for each eigen space of A.

(ii) Show that A is diagonalizable and find the exact relation between A , P and D.
(Do **not** calculate P^{-1}).

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4. (9 %) Let T: V→W be a linear transformation of vector spaces. If $T(a_1), T(a_2), \ldots, T(a_n)$ are linearly independent and dimV=n, show that $\{a_1, a_2, \ldots, a_n\}$ is a basis of V.

5. (9 %) Let T: V→W be a linear transformation of vector spaces. (i) State the rank-nullity theorem for T, (ii) then use it to show that if T is onto and dimV=dimW=n, then T must be is injective.

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6. (9 %) Let $A=\begin{bmatrix}1&2&1&6&3&8\\2&4&0&2&2&2\\0&0&1&5&2&7\end{bmatrix}$

(i) Show that rank A=2

(ii) Does the system $AX=B$ have a solution for every B in R^3? **Justify.**

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7. (9 %) Let $\{b_1, b_2, \ldots, b_n, a_1, a_2, \ldots, a_m\}$ be an o.n basis of an inner product space V.
Let B= span $\{b_1, b_2, \ldots, b_n\}$ & A=span $\{a_1, a_2, \ldots, a_m\}$. Show that

(i) $V=A\oplus B$ (ii) $B=A^{\perp}$.

8. (5 %) For any symmetric $n\times n$ matrix A, show that A and A^5 have the same null space.