AUB
Mathematics 218
Fall: 2002-2003

Final Exam
Time : 2h.

I- Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 3 & -1 \end{bmatrix}$

1) Calculate A^2 and A^3 and show that $A^3 = 9A - 8I_3$

2) Deduce that A is invertible and Find A^{-1}

(10pts)

II- 1) Use the rank to determine the values of k for which the linear system

$$\begin{aligned} x + k\cdot y &= 1 \\ kx + y &= -1 \end{aligned}$$

has no solution, one solution or infinitely many solutions.

2) For $k = 1$, find the least square solution(s) of the linear system

(15pts)

III- Let $V = R^3$ be the Euclidean inner product and $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix}$

and let W be the Null space of A.
1) Find rank (A).
2) Find a basis for W and a basis for $W^{\perp}$
3)Let $v = (1, 1, 1)$. Find the orthogonal projection of v on W.

(15pts)

IV- Let $V = M_{n\times n}$ be the vector space of all $n \times n$ matrices.
Let C be a fixed $n \times n$ invertible matrix and $T : V \to V$ be the function defined by $T(A) = CA$ for all A in $M_{n \times n}$
1) Show that T is a linear operator
2) Show that T is one - to - one and deduce $\text{rank}(T) = n^2$

(10pts)

V- Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

1) Show that the characteristic equation of A is $\lambda^2 (\lambda - 3) = 0$
2) Find an orthonormal basis of every eigenspace corresponding to every eigenvalue
3) Find a matrix P that orthogonally diagonalizes A.

(20pts)

VI- Prove or disprove (by a counter example) the following:

1) Every square matrix with a non zero determinant has a QR-decomposition.

2) Let A, B be row equivalent $n \times n$ matrices.

   a) There exists an invertible $n \times n$ matrix C such that $B = CA$

   b) A is invertible if and only if B is invertible

   c) A and B have the same Null space.

   d) A and B have the same column space

(15pts)

VII- Let V be an inner product space and let $T : V \to V$ be a linear operator.

Suppose that $\| T(v) \| = \| v \|$ for all v in V.

1) Show that T is one – to – one .

2) Show that $< T(u), T(v) > = < u , v >$ for all u , v in V
   (Hint: Use $\| T(u+v) \|^2 = \| u+v \|^2$)

3) Suppose $\dim(V) = 2$ and let $B = \{ v_1, v_2 \}$ be an orthonormal basis of V . Let $B_1 = \{ T(v_1) , T(v_2) \}$

   a) Show that B_1 is an orthonormal set of V

   b) Show that B_1 is a linearly independent set of V.

   c) Deduce that B_1 is an orthonormal basis of V .

(15pts)
