AMERICAN UNIVERSITY OF BEIRUT

Mathematics Department

Math 218 - Final Exam

Fall 2005-2006

Name:................................. **ID:**...............................

Section 1	**Section 2**	**Section 3**
Mrs. Z. Mouzeihem	**Ms. M. Houri**	**Ms. D. Audi**

Time: 120 min

I- (**35 points**) Let

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 7 & 2 & 2 \\ 2 & 4 & 2 & 2 \end{pmatrix}$$

(a) Find a basis for the row space of A.

(b) Find a basis for the column space of A.

(c) Find a basis for the null space of A.

(d) Find a basis for the null space of A^T.

(e) If $W^\perp$ is the column space of A deduce a basis for W.

II- (**25 points**) Let $T : P_2 \to P_2$ be a linear transformation such that:

$$T(p(x)) = p(2x + 3)$$

a) Find the matrix $A = [T]_B$ with respect to the standard basis $B = \{1, x, x^2\}$.

b) Find the eigenvalues for the matrix $A = [T]_B$

c) Write the formula for $[T(p(x))]_B$ without justification.

d) Deduce the values of λ such that $T(p(x)) = \lambda p(x)$ has nonzero solutions.

III- (**20 points**) Let $S = \{u_1, u_2, u_3\}$ be a basis for $\mathbf{R}^3$, with $u_1 = (2, 2, 0)$, $u_2 = (2, 0, 0)$ and $u_3 = (0, 0, 3)$

a) Find an orthonormal basis $S_1 = \{q_1, q_2, q_3\}$ for $\mathbf{R}^3$.

b) Write $u = (1, 1, 1)$ in terms of the basis S_1.

IV- (**20 points**) Let $T : \mathbf{R}^3 \to \mathbf{R}^4$ be defined by:

$$T(x, y, z) = (x + 2y, x + 3z, z - x, 0)$$

a) Show that T is a linear transformation.

b) Is T one-to-one? Justify

c) Find the rank of T.

d) Find a basis for the range of T.

V- (**25 points**) Let A be a 3×3 matrix with eigenvalues $\lambda_1 = -2$, $\lambda_2 = 0$ and $\lambda_3 = 4$, and let v_1, v_2 and v_3 be the corresponding eigenvectors respectively, where $v_1 = (0, 3, -2)$, $v_2 = (-3, -3, 0)$ and $v_3 = (-2, 0, 3)$.

Let $E_1 = \text{span}\{v_1\}$ be the eigenspace corresponding to $\lambda_1 = -2$, $E_2 = \text{span}\{v_2\}$ the eigenspace corresponding to $\lambda_2 = 0$ and $E_3 = \text{span}\{v_3\}$ the eigenspace corresponding to $\lambda_3 = 4$.

a) Write $x = (-10, -3, 4)$ as a linear combination of the vectors $\{v_1, v_2, v_3\}$, and deduce Ax.

b) Find the null space of A (*Hint:* Use the fact that $\lambda = 0$ is an eigenvalue for A.)

c) Find a matrix P that diagonalizes A and determine $P^{-1}AP$

VI- (15 points) Short Proofs

1) Show that if A is an orthogonal matrix then $\det(A) = \pm 1$.

2) State and prove the theorem of Pythagoras for an inner product space V.

3) Let $T : V \to W$ be a linear transformation, show that $\mathrm{Ker}(T)$ is a subspace of V.

VII- (**30 points**) Show your work.

a) Let A, B and C be $n \times n$ matrices, if $B = A^{-1}$and $C = B^T$ find C^{-1}.

b) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear operator with $T(1, 1) = (2, 1)$ and $T(3, 2) = (0, 3)$, find $T(5, 4)$.

c) Let $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 5 & 2 \\ 1 & 0 & 8 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$.
If B is a 3×3 matrix and $AB = C$ find $\det(B)$.

d) Let $A = \begin{pmatrix} 4 & 1 \\ d & -4 \end{pmatrix}$, find the value of d that makes A orthogonally diagonalizable.

e) Find the null space of a 5×4 matrix with linearly independent columns.

f) Let $\{u, v, w\}$ be a basis for a vector space V. Show that $\{w, u + w, u + v + w\}$ are linearly independent.

VII- (**30 points**) Answer TRUE or FALSE **without proof** (**5 points** for each right answer and **2 points** penalty for each wrong answer).

a- Let A be an $n \times n$ matrix, if $\lambda = 0$ is an eigenvalue for A then A is invertible.

b- Let $T : P_n \to P_n$ be a linear transformation, if range of T is P_n then T is one-to one.

c- Let A, B and C be $n \times n$ matrices, if $AB = AC$ then $B = C$.

d- Let A be an $m \times n$ matrix and b be an $m \times 1$ vector, if b is not in the column space of A then $AX = b$ has no solution.

e- Let $T : V \to \mathbf{R}$ be a linear transformation with $V = \text{span}\{u\}$. If $T(u) = 0$ then $T \equiv 0$.

f- Cramer's rule can be used to solve any system of linear equations.

GOOD LUCK!!