AMERICAN UNIVERSITY OF BEIRUT

Mathematics Department

Math 218 - Final Exam

Fall 2007-2008

Name:..

ID:...

Section: 4 (@ 12:30) 5 (@ 9:30)

Time: 120 min

I- Given the vector space $\mathbb{R}^4$ with standard addition and scalar multiplication.

Let $W = \{x \in \mathbb{R}^4 \text{ such that } Ax = 0\}$ where $A = \begin{bmatrix} 3 & 6 & 0 & 0 \\ 3 & 7 & 0 & -15 \\ 2 & 4 & 0 & 0 \end{bmatrix}$.

(a) (i) **(4 points)** Show that W is a subspace of $\mathbb{R}^4$.

(ii) **(5 points)** Find a basis for W.

(iii) **(1 point)** Determine the dimension of W.

(b) **(6 points)** Find a basis for $W^{\perp}$, the orthogonal complement of W, with respect to the Euclidean inner product.

II- Given $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 6 & 7 \\ 3 & 0 & 8 & 9 \\ 4 & 0 & 0 & 10 \end{bmatrix}$

(a) **(3 points)** Find the eigenvalues of A.

(b) **(8 points)** Find bases for each of the eigenspaces of A.

Continue ...

(c) **(2 points)** Explain, *without any calculation*, whether the column vectors of A are a basis for $\mathbb{R}^4$.

(d) **(1 point)** Give *without justification* a matrix P such that $P^{-1}AP$ is a diagonal matrix. (That is P diagonalizes A)

(e) **(2 points)** If A is the standard matrix representing a linear transformation T and we take M_{44} with the inner product $< A, B >= tr(A^T B)$. Is the function f defined by $f(x) = \|A\| \, T(x)$ a linear transformation? Justify.

III- (5 points) ***Choose one*** of the following statements and prove it.

- Given an $m \times n$ matrix A, if $Ax = b$ is consistent for all b then the column vectors of A span $\mathbb{R}^m$.
- If a set of nonzero vectors in an inner product space V is orthogonal then it is linearly independent.

IV- (a) **(8 points)** Find a least squares solution for the system:

$$\begin{cases} x & & -z = 6 \\ 2x & +y & -2z = 27 \\ 3x & +3y & = 0 \\ x & +y & -z = 3 \end{cases}$$

(b) **(4 points)** Use part (a) to find $Proj_W b$ where

$$b = \begin{bmatrix} 6 \\ 27 \\ 0 \\ 3 \end{bmatrix} \text{ and } W = Span\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ -1 \end{bmatrix} \}$$

V- (3 points for each part) Give a short answer.

1- The nullity of $A = \begin{bmatrix} 3 & 0 & 1 & 2 & 5 \\ 0 & 1 & 5 & 2 & -1 \\ 3 & 0 & 5 & -6 & 4 \end{bmatrix}$ is:

2- The vector space of all diagonal $n \times n$ matrices has dimension:

3- If $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where
$T_1(e_1) = (1, 2)$
$T_1(2e_2) = (0, 6)$
$T_2(e_1) = T_1(e_1 + e_2)$
$T_2(e_2) = -3T_1(2e_1)$
then $T_1 \circ T_2(2, -1)$ is:

4- If $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the projection on the x-axis then the standard matrix representing T is:

5- An orthogonal basis for the space spanned by $u_1 = (1, 0, 3, 2)$ and $u_2 = (-2, 1, 0, 3)$ with respect to the Euclidean inner product is:

VI- (3 points for each part) Use the given to detect the unknown. Justify.

1- A is a matrix where the number of columns is twice the number of rows. The system $Ax = b$ is consistent for all b. The column vectors of A span a subspace of $\mathbb{R}^6$.
What is $rank(A)$?

2- A is an $n \times n$ matrix with row vectors $r_1, r_2, ..., r_n$ and column vectors $c_1, c_2, ..., c_n$. The set $\{r_1, r_2, ..., r_n, c_1, c_2, ..., c_n\}$ is linearly independent.
What is $ColumnSpace_A$?

3- S is a set of matrices all of which represent linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$. All linearly independent subsets of S with 8 elements will not span S.

What is S?

VII- (2 points for each part) Answer **TRUE** or **FALSE**. **DO NOT JUSTIFY YOUR ANSWER**. Ambiguous answers will not be counted.

1- The span of a set of 6 nonzero vectors in any vector space V has infinitely many elements.

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2- $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $T(x, y, z) = (3x + y, z - 3)$ is a linear transformation.

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3- If V is any inner product space and u and v are any two orthogonal elements of V then $\|u\|^2 + \|v\|^2 = \|u + v\|^2$.

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4- The function $f(x) = mx + b$ is a linear transformation from $\mathbb{R}$ to $\mathbb{R}$ for any two scalars m and b.

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5- Given any inner product space V and any two elements u and v of V we have $-\|u\|\|v\| \leqslant < u, v > \leqslant \|u\|\|v\|$.

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6- If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation and $m = n$ then T is one-to-one.

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7- If k is a scalar and u is an element of a vector space V where $ku = 0$ then either $k = 0$ or $u = 0$, BUT we can have two nonzero matrices A and B where $AB = 0$.

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8- $\begin{bmatrix} 5 & 4 & 5 \\ 7 & -5 & 9 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 6 \end{bmatrix}$, $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 5 & -3 \end{bmatrix}$, and $\begin{bmatrix} 6 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$

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9- The set of all 2×2 matrices A with $det(A) = 0$ is a subspace of M_{22}.

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10- If S_1 and S_2 are two subsets of a vector space V and $Span(S_1) = Span(S_2)$ then $S_1 = S_2$.

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11- Any subset of a linearly independent set is linearly independent.

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12- If $\{v_1, v_2, v_3, v_4\}$ is a basis for a vector space V, then $\{3v_1, 5v_2, -v_3, -v_4\}$ is also a basis for V.

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13- An upper triangular matrix of size 5×5 has exactly 5 distinct eigenvalues.

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14- For any $m \times n$ matrix A, the associated normal system is either $A^T Ax = A^T b$ or $AA^T x = A^T b$ since $A^T A$ is always symmetric for any A.

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GOOD LUCK