Math.218. Final (120 minutes). June2004. R. Barbara.

1 Answer by true or false: _Every subset of a linearly independent set is a linearly independent set.

_ The vectors (1, 0, 1), (1, 1, 0), (3, 2, 0) form a basis for $\mathbb{R}^3$.

_The column vectors of a 7×4 matrix must be linearly dependent.

_If span(S) = span(T), then, S = T.

_If {a, b, c} is a linearly independent set of nonzero vectors, then, each is a linear combination of the others.

_If u, v are vectors of a inner product space, then, $\|u\| = \|v\|$ *if and only if* u − v is orthogonal to u + v.

2 If we modify the scalar multiplication in $\mathbb{R}^2$ by setting k(x,y) = (2kx, 2ky), does $\mathbb{R}^2$ remain a vector space ?

3 Is $W = \{ (x, y, z) \in \mathbb{R}^3 ,\ y = x + z + 1 \}$ a sub-vector space of $\mathbb{R}^3$?

4 Prove or disprove that x = (3, 1, 5) is a linear combination of u = (0, −2, 2) and v = (1, 3, −1).

5 Does the solution set of the linear system $x + y + z = 0$, $x - 2y + z = 1$ form a sub-vector space of $\mathbb{R}^3$?

6 Determine whether the vectors u = (2, −1, 3), v = (4, 1, 2), w = (0, 3, −4) form a basis for $\mathbb{R}^3$.

7 Are the vectors a = (0, 0, 1, 2), b = (5, 1, 2, 0), and c = (1, 0, 0, 3) linearly independent in $\mathbb{R}^4$?

8 Find the coordinate vector of $\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ relative to the basis $A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

9 Find a basis for the nullspace of A, and the rank of A, where $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$.

10 Compute <u, v>, where < > denotes the inner product on $\mathbb{R}^3$ generated by the matrix $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 5 \end{bmatrix}$.

11 Consider the weighted inner product on $\mathbb{R}^2$ defined by $\langle u, v \rangle = 3.u_1 v_1 + 5.u_2 v_2$.

- Find the norm of u = (−8, 15).
- Find the distance between u and v = (17, −3).
- Find the cosine of the angle between u and v.

12 Let T be the linear transformation: $\mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(x, y, z, t) = (40x - 20z + 5t,\ \ x + y + z + t,\ \ 6t)$.

Find a basis for the Kernel of T. *Subsidiary question*: Is it possible to modify the numerical values above in order to obtain rank(T) = 4 ?

13 Let n be a positive integer, and let $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$. Find A^n.

14 Let t denote a real parameter. Consider the 3×3 matrix $A = \begin{bmatrix} t/3 & 2t/3 & 2t/3 \\ 2t/3 & -2t/3 & t/3 \\ -2t/3 & -t/3 & 2t/3 \end{bmatrix}$. Find all the values of t

for which matrix A is orthogonal.(a complete proof is needed).

15 Prove or disprove: We can find a 1×3 matrix M such that, *for infinitely many* 3×1 vectors v, Mv is a prime number.

16 Let p and q be real numbers and suppose that the equation $x^3 + p.x^2 + x + q = 0$ has at least one real zero ε. Prove the existence of a 4×4 matrix A satisfying $A^3 + A = p.A^2 + q.I_4$.

17 The following property is well-known: If A is a m×n matrix with rank n, then, $A^T A$ is invertible.

Does the result remain true if we allow complex entries ?

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