Name: Signature: Student Number:

Mathematics 218, Sections 1 and 2

Final Examination, June 13, 2005, 8:00–10:00

Important Instructions:

1. This exam consists of two sets of problems. The ten "TRUE/FALSE answer" questions are numbered A, B, ..., J; each of them is worth 4 pts. The eight "workout" questions are numbered $1, 2, \ldots, 8$; each of them is worth 8 pts. The maximum score is 104 points.

2. Do not separate these pages.

3. The pink booklets are merely a source of scrap paper. I will not read what is in them.

If you have any comments/requests write them here:

No calculators.

Good Luck!

Part I.

Write your answer TRUE or FALSE under each of the following problems/statements:

Problem A. If $A = [a_{ij}]_{m\times n}$ has linearly independent row vectors, then $\text{rank}(A) = n$.

Your answer:

Problem B. If A is an orthogonal matrix, then $A^{-1} = A^T$.

Your answer:

Problem C. An invertible matrix can have complex eigenvalues.

Your answer:

Problem D. Two planes in $\mathbb{R}^5$ can have exactly one common point.

Your answer:

Problem E. If $\vec{u}$ is an eigenvector of a matrix A, then $-3\vec{u}$ is also an eigenvector of A.

Your answer:

Problem F. A consistent system of 5 linear equations with 7 unknowns must have infinitely many solutions.

Your answer:

Problem G. V and W are vector spaces, and $T : V \to W$ is a linear transformation. If $\ker(T) \neq \{\vec{0}\}$, then T is one-to-one.

Your answer:

Problem H. If A and B are square matrices of the same size and A is singular, then AB can be invertible.

Your answer:

Problem I. $A\vec{x} = \vec{b}$ is a system of linear equations. If A and $[A|\vec{b}]$ have the same rank, then the system is inconsistent.

Your answer:

Problem J. Each singular matrix can be diagonalized.

Your answer:

Part II.

You must provide all essential details of your solution to each of the following problems $1, \ldots, 8$ on the page that contains the problem (continue on the reverse side of that page when needed).

Problem 1.

Use the Gram-Schmidt process to orthogonalize the following set of 3 vectors in $\mathbb{R}^5$:
$\vec{v}_1 = (1,0,1,0,1)$, $\vec{v}_2 = (1,1,1,0,0)$, $\vec{v}_3 = (0,0,1,1,1)$ (do not change their order of appereance).

Problem 2.

Find a matrix P that diagonalizes the following matrix A. Then find $P^{-1}AP$.

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

Problem 3.

Find the least squares solution of the linear system

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Problem 4.

Prove that if λ is an eigenvalue of a square matrix $B = [b_{ij}]_{n\times n}$, then λ^4 is an eigenvalue of B^4.

$$B = P^{-1}DP$$

$$D = PBP^{-1}$$

$$D^2 = (P^2 B^2 P^{-2})^2$$

$$D^2 = (PBP^{-1})^2 = (PBP^{-1})(PBP^{-1})$$

$$D^2 = PBIBP^{-1} = PB^2P^{-1}$$

$$D^k = PB^kP^{-1}$$

$$\lambda^k \lambda^k \lambda^k$$

Problem 5.

Let V and W be vector spaces, $T : V \to W$ be a linear transformation, and $S = \{v_1, v_2, \ldots, v_k\}$ be a linearly independent set of vectors in V.

Prove that if T is one-to-one, then $\{T(v_1), T(v_2), \ldots, T(v_k)\}$ is a linearly independent set of vectors in W.

Problem 6.

Let A be a matrix of size $m \times n$. Prove that AA^T is always a symmetric matrix.

Problem 7.

Let $A = [a_{ij}]_{n \times n}$ be a lower triangular matrix. Write a proof that $\det(A) = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$.

Problem 8.

Write a detailed explanation how the angle between two non-zero vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is defined and why this definition makes sense.