Name: **Signature:** **Student Number:**

Mathematics 218
Final Exam, August 12, 2004, 8:00 – 10:00

Important Instructions:
1. This exam consists of 26 problems. They are divided into 2 parts. The maximum score is 102 points.
2. Only white booklets will be graded.
3. The pink booklets are merely a source of scrap paper. I will not read what is in them. You may take your pink booklet after the exam.
4. No calculators.

Good Luck!

Part I. Write your answer **TRUE** or **FALSE** under each of the following problems/claims 1-20. Each correct answer is worth 3 pts.

Problem 1. Each non-zero subspace of $\mathbb{R}^n$ has an orthonormal basis.

Your answer (write TRUE or write FALSE):

Problem 2. Let $A = \begin{pmatrix} 3/7 & -6/7 & 2/7 \\ 2/7 & 3/7 & 6/7 \\ 6/7 & 2/7 & -3/7 \end{pmatrix}$ Then A is an orthogonal matrix.

Your answer (write TRUE or write FALSE):

Problem 3. $\lambda = 2$ is an eigenvalue of the matrix $\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Your answer (write TRUE or write FALSE):

Problem 4. If $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\|\vec{u}\| = 3$ and $\|\vec{v}\| = 4$, then $\vec{u} \cdot \vec{v} \leq 6$.

Your answer (write TRUE or write FALSE):

Problem 5. rank$(A) = 3$ where $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

Your answer (write TRUE or write FALSE):

Problem 6. V is a vector space and we have three linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in V$. Let $\vec{w}_1 = \vec{v}_1 - \vec{v}_2 + 2\vec{v}_3$, $\vec{w}_2 = 2\vec{v}_1 + \vec{v}_2 - \vec{v}_3$ and $\vec{w}_3 = \vec{v}_1 + 5\vec{v}_2 - 8\vec{v}_3$. Then the vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$ are linearly independent, too.

Your answer (write TRUE or write FALSE):

Problem 7. We have vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n$. If $m > 2n$, then these vectors span $\mathbb{R}^n$.

Your answer (write TRUE or write FALSE):

Problem 8. Let P denote the vector space of all polynomials, and let W be the set of all polynomials $p(x)$ such that degree$(p(x)) \leq n$ and $p(3) = 0$. Then W is a subspace of P.

Your answer (write TRUE or write FALSE):

Problem 9. Each basis for an n-dimensional vector space V must consist of exactly n vectors.

Your answer (write TRUE or write FALSE):

Problem 10. A is a square matrix of size $n \times n$. If rank$(A) \neq n$, then A is singular.

Your answer (write TRUE or write FALSE):

Problem 11. V is a vector space and V_1 and V_2 are its subspaces. Then their union $V_1 \cup V_2$ is also a subspace of V.

Your answer (write TRUE or write FALSE):

Problem 12. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. If T is one-to-one, then the range of T is $\mathbb{R}^n$.

Your answer (write TRUE or write FALSE):

Problem 13. If $\vec{u}$ is an eigenvector of a matrix A, then $-2\vec{u}$ is also an eigenvector of A.

Your answer (write TRUE or write FALSE):

Problem 14. Two planes in $\mathbb{R}^4$ can have exactly one common point.

Your answer (write TRUE or write FALSE):

Problem 15. Let $\vec{u}_1 = (1,0,0,0)$, $\vec{u}_2 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)$, $\vec{u}_3 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0)$ and $\vec{u}_4 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Then $S = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is an orthonormal basis for $\mathbb{R}^4$.

Your answer (write TRUE or write FALSE):

Problem 16. If A is a matrix which is not square, then the column vectors of A are linearly dependent.

Your answer (write TRUE or write FALSE):

Problem 17. If A and B are invertible matrices of the same size, then $\det(AB) = \det(BA)$.

Your answer (write TRUE or write FALSE):

Problem 18. A consistent system of linear equations must have infinitely many solutions.

Your answer (write TRUE or write FALSE):

Problem 19. $\det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 2 & -1 \\ 1 & 3 & 2 & 0 \end{pmatrix} = 5$

Your answer (write TRUE or write FALSE):

Problem 20. If A is an orthogonal matrix, then A^2 is an orthogonal matrix, too.

Your answer (write TRUE or write FALSE):

Part II. Give detailed solutions/proofs for Problems 21 – 26. Continue on the reverse side of the problem's page when needed.

Problem 21. 6 pts.

$T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation such that $T(1,2,3) = (1,1,1,1)$, $T(1,3,4) = (1,1,1,0)$ and $T(1,4,4) = (0,1,1,1)$. Find the standard matrix A_T for T.

Your solution:

Problem 22. 6 pts.

Find the least squares solution of the linear system $\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Your solution:

Problem 23. 6 pts.

A is an $m \times n$ matrix which has linearly independent column vectors. Prove that the system $A\vec{x} = \vec{0}$ has the trivial solution only.

Your proof:

Problem 24. 6 pts.

A and B are square matrices of size $n \times n$. Prove that if A is invertible, then the row spaces of AB and B are equal.

Hint: Each invertible matrix is a product of elementary matrices.

Your proof:

Problem 25. 6 pts.

Determine whether multiplication by the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$ is a one-to-one transformation of the appropriate Euclidean spaces.

Your solution:

Problem 26. 12 pts.

Find a basis for the row space, a basis for the column space, and a basis for the nullspace of the matrix $A = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 3 & 0 & 2 & 2 \\ 1 & 2 & -2 & -4 \end{pmatrix}$

Your solution: