AMERICAN UNIVERSITY OF BEIRUT

Mathematics Department

Math 218 - Quiz I

Spring 2006-2007

Name: ..Solution.................. **ID:**......................................

Section: **9 (@ 9:30)** **2 (@ 11:00)** **1 (@ 12:30)**

Time: 60 min

This is a partial assessment of what we discussed. My only aim is to see what you learned and detect the problems you may have (if any) so that we can together correct them. Work quickly but accurately. Take a deep breath and show me that you know!!

I- (15 points) Solve the following system by using either Gaussian or Gauss-Jordan elimination.

$$\begin{cases} x_1 + 4\,x_2 - 3\,x_3 + x_4 = 5 \\ 8\,x_2 + 2\,x_1 - 10\,x_3 + 4\,x_4 = 24 \\ x_3 + x_4 = 2 \\ x_1 + 4\,x_2 - 5\,x_3 + 2\,x_4 = 12 \\ -3\,x_1 - 12\,x_2 + 17\,x_3 - 4\,x_4 = -32 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & -3 & 1 & 5 \\ 2 & 8 & -10 & 4 & 24 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & 4 & -5 & 2 & 12 \\ -3 & -12 & 17 & -4 & -32 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 4 & -3 & 1 & 5 \\ 0 & 0 & -4 & 2 & 14 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & 1 & 7 \\ 0 & 0 & 8 & -1 & -17 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 4 & -3 & 1 & 5 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 6 & 22 \\ 0 & 0 & 0 & 3 & 11 \\ 0 & 0 & 0 & -9 & -33 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 4 & -3 & 1 & 5 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 3 & 11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

continue your answer here:

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 4 & -3 & 1 & 5 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 3 & 11/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

10

$\boxed{x_4 = \frac{11}{3}}$

5

$x_3 + x_4 = 2$

$\boxed{x_3 = 2 - \frac{11}{3} = \frac{-5}{3}}$

$\boxed{x_2 = t}$

$x_1 = 5 - x_4 + 3x_3 - 4x_2$

$= 5 - \frac{11}{3} - 5 - 4t$

$= -\frac{11}{3} - 4t$

so $(x_1, x_2, x_3, x_4) = \left(-\frac{11}{3} - 4t, t, \frac{-5}{3}, \frac{11}{3}\right) \quad t \in \mathbb{R}$

II- Consider the following matrices,

$$A = \begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 26 & 0 & 1 & 0 \\ 26 & 0 & 1 & 26 \\ 1 & 5 & 0 & -1 \end{bmatrix}$$

(a) **(6 points)** Find A^{-1}.

$\det(A) = 1 \begin{vmatrix} \sqrt{2} & 3\sqrt{2} \\ -4\sqrt{2} & \sqrt{2} \end{vmatrix} = 2 + 12 \times 2 = 26$

$$A^{-1} = \frac{1}{26}\begin{bmatrix} \sqrt{2} & 4\sqrt{2} & 0 \\ -3\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 26 \end{bmatrix}^T = \begin{bmatrix} \frac{\sqrt{2}}{26} & \frac{-3\sqrt{2}}{26} & 0 \\ \frac{4\sqrt{2}}{26} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) **(9 points)** Find B if $AB^T = C$.

$AB^T = C \Rightarrow B^T = A^{-1}C$

$$B^T = \begin{bmatrix} \frac{\sqrt{2}}{26} & \frac{-3\sqrt{2}}{26} & 0 \\ \frac{4\sqrt{2}}{26} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 26 & 0 & 1 & 0 \\ 26 & 0 & 1 & 26 \\ 1 & 5 & 0 & -1 \end{bmatrix}$$

continue your answer here:

$$= \begin{bmatrix} -2\sqrt{2} & 0 & \frac{-2\sqrt{2}}{26} & -3\sqrt{2} \\ 5\sqrt{2} & 0 & \frac{5\sqrt{2}}{26} & \sqrt{2} \\ 1 & 5 & 0 & -1 \end{bmatrix}$$

so $$B = \begin{bmatrix} -2\sqrt{2} & 5\sqrt{2} & 1 \\ 0 & 0 & 5 \\ \frac{-\sqrt{2}}{13} & \frac{5\sqrt{2}}{26} & 0 \\ -3\sqrt{2} & \sqrt{2} & -1 \end{bmatrix}$$

III- Consider an $n \times n$ matrix A.

(a) **(7 points)** Show that $tr(A) = tr(A^T)$.

$A = [a_{ij}]_{n \times n}$

$tr(A) = a_{11} + a_{22} + --- a_{nn}$

$A^T = [a'_{ij}]_{n \times n} = [a_{ji}]_{n \times n}$

$tr(A^T) = a_{11} + a_{22} + -- a_{nn}$

so $tr(A) = tr(A^T)$

(b) **(7 points)** Show that if A is a symmetric and invertible matrix, then A^2 is also symmetric and invertible.

Given: $A = A^T$

want: $(A^2)^T = A^2$

$$A^2 = A.A = A^T.A^T = (A.A)^T = (A^2)^T$$

want: A^2 invertible (i.e $(A^2)^{-1}$ exists)

We will prove that $A^{-1}.A^{-1}$ is $(A^2)^{-1}$

$$(A^{-1}.A^{-1}) = (A.A)^{-1} = (A^2)^{-1}.$$

so A^2 is invertible.

IV- **(a)** If A and B are 5×5 matrices such that $det(A) = 2$ and $det(B) = 3$, find:

(i) **(6.5 points)** $det(A\,adj(A))$

$$A^{-1} = \frac{1}{det(A)} adj(A) \quad \text{or} \quad A.adj(A) = det(A)\, I$$

$$det(A.adj(A)) = det(det(A)\, I)$$
$$= (det(A))^5 \times det(I) = 2^5.$$

(ii) **(6.5 points)** $det(A^2BA^{-1})$

$$det(A^2 B A^{-1}) = det(A).det(A).det(B).det(A^{-1})$$
$$= 2 \times 2 \times 3 \times \frac{1}{2}$$
$$= 6$$

(b) **(5 points)** If A is an $n \times n$ invertible matrix, is it possible to have $det(A^k) = det(kA)$ for some $k \geqslant 2$ if also $det(A) \neq 1$? Justify.

$det(A^k) = det(A)^k$

$det(kA) = k^n \, det(A)$

$det(A)^k = k^n \, det(A)$ 2

Say $det(A) = x$, $x \neq 0$ 0.5

so $x^k = k^n x$

$x^{k-1} = k^n$ $\quad x = k^{\frac{n}{k-1}}$

Take $n = 4$ $\quad k = 5$

So for example take $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

(c) **(6.5 points)** Assuming that the stated inverses exist, show that, $BA(C-A)^{-1}$ and $C(A^{-1} - C^{-1})B^{-1}$ are inverses of each other.

$$\left(BA(C-A)^{-1}\right)^{-1} = (C-A)(BA)^{-1}$$
$$= (C-A)A^{-1}B^{-1} = CA^{-1}B^{-1} - B^{-1}$$

But $C(A^{-1} - C^{-1})B^{-1} = CA^{-1}B^{-1} - CC^{-1}B^{-1}$

$= CA^{-1}B^{-1} - B^{-1}$

So, they are inverses.

V- (10 points) Prove the following identity without directly computing the determinant.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+(t-1)c_1 & b_1-a_1^2 & c_1-\ b_1^2+\ a_1^2b_1 \\ a_2+(t-1)c_2 & b_2-a_1a_2 & c_2-\ b_1b_2+\ a_1a_2b_1 \\ a_3+(t-1)c_3 & b_3-a_1a_3 & c_3-\ b_1b_3+\ a_1a_3b_1 \end{vmatrix}.$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \overset{\text{transpose}}{=} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\xrightarrow{-a_1 Col_1 + Col_2} = \begin{vmatrix} a_1 & b_1-a_1^2 & c_1 \\ a_2 & b_2-a_1a_2 & c_2 \\ a_3 & b_3-a_1a_3 & c_3 \end{vmatrix} \xrightarrow{(t-1)Col_3+Col_1} = \begin{vmatrix} a_1+(t-1)c_1 & b_1-a_1^2 & c \\ a_2+(t-1)c_2 & b_2-a_1a_2 & c \\ a_3+(t-1)c_3 & b_3-a_1a_3 & c_3 \end{vmatrix}$$

$$\xrightarrow{-b_1 Col_2+Col_3} = \begin{vmatrix} a_1+(t-1)c_1 & b_1-a_1^2 & c_1-b_1(b_1-a_1^2) \\ a_2+(t-1)c_2 & b_2-a_1a_2 & c_2-b_1(b_2-a_1a_2) \\ a_3+(t-1)c_3 & b_3-a_1a_3 & c_3-b_1(b_3-a_1a_3) \end{vmatrix}$$

$$= \begin{vmatrix} a_1+(t-1)c_1 & b_1-a_1^2 & c_1-b_1^2+a_1^2b_1 \\ a_2+(t-1)c_2 & b_2-a_1a_2 & c_2-b_1b_2+a_1a_2b_1 \\ a_3+(t-1)c_3 & b_3-a_1a_3 & c_3-b_1b_3+a_1a_3b_1 \end{vmatrix}$$

VI- a- **(15 points)** YOU **DO NOT** NEED TO GIVE REASONS FOR YOUR ANSWERS IN THIS PART. Answer by **TRUE** or **FALSE**.

i- If a matrix A can not be written as a product of elementary matrices, then the product AB is not invertible for any matrix B. T

ii- If B is any square matrix, then $B^{-k} = \frac{1}{B^k}$ for any positive integer k. F

iii- If A and B are row equivalent matrices, then either both are invertible or both are not invertible. T

iv- A homogeneous linear system is always consistent. T

v- An upper triangular matrix with a zero entry on the main diagonal can never have a nonzero elementary product. T

b- **(6.5 points)** Give an example of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that $\|T(1,1)\| = 3$ and explain why it is a linear transformation.

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(x,y) \longrightarrow (2x, 2y, x)$$

matrix representing it is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$

so it is a linear transformation

$$\|T(1,1)\| = \sqrt{2^2+2^2+1^2} = \sqrt{9} = 3$$

GOOD LUCK