**Problem 1**

(18 pts) Use the curl test to show that the differential form in the following integral is exact (meaning, the corresponding vector field is conservative). Then evaluate the integral.

$$\int_{(0,1,1)}^{(1,2,1)} \left(2xyz + x\cos(x^2)\right) dx + \left(x^2 z\right) dy + \left(x^2 y\right) dz$$

$$\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2xyz + x\cos(x^2) & x^2 z & x^2 y \end{vmatrix}$$

$F(2xyz + x\cos(x^2),\ x^2 z,\ x^2 y)$

$= \vec{i}\,(x^2 - x^2) - \vec{j}\,(2xy - 2xy) + \vec{k}\,(2xz - 2xz)$

$= 0\vec{i} - 0\vec{j} + 0\vec{k} = 0$ so $\vec{F}$ is conservative

so $F$ has a potential function $f$ s.t. $\nabla f = F$

hence:

$$\int_{(0,1,1)}^{(1,2,1)} (2xyz + x\cos(x^2))\,dx + (x^2 z)\,dy + (x^2 y)\,dz$$

$= f(1,2,1) - f(0,1,1)$

looking for $f$:

$\nabla f\,(2xyz + x\cos x^2,\ x^2 z,\ x^2 y)$

- so $f_x = 2xyz + x\cos x^2$

  $f = \int (2xyz + x\cos(x^2))\,dx = x^2 yz + \frac{1}{2}\sin x^2 + g(y,z)$

- $f_y = x^2 z$ & $f_y = x^2 z + g_y(y,z) + 0$

  hence $g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$

- $f_z = x^2 y$ & $f_z = x^2 y + 0 + h_z(z)$

  so $h_z(z) = 0 \Rightarrow h(z) = C$

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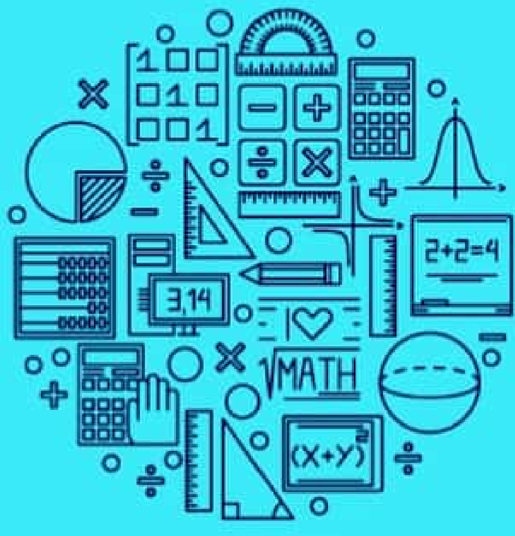
# MATH TUTORS !

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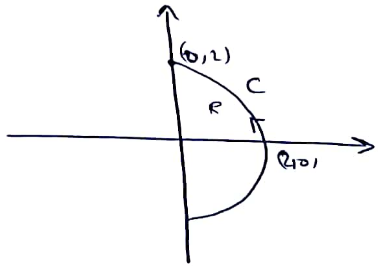
**Problem 2**

(18 pts) Let $R$ be the semi-circular region in the $xy$-plane bounded from the left by the $y$-axis and from the right by the circle $x^2 + y^2 = 4$.
Let $C$ be the curve bounding this region.
Use Green's theorem to find the counterclockwise circulation and outward flux of the vector field

$\mathbf{F} = \left(x^2 y + x\right)\mathbf{i} - \left(x y^2\right)\mathbf{j}$ around and across $C$.




Circulation $= \iint_R (N_x - M_y)\,dA$ (G.T)

$N_x = -y^2$

$M_y = x^2$

$= \iint_R -y^2 - x^2 \, dA$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 -r^2 \cdot r\,dr\,d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 -r^3\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-\frac{r^4}{4}\right]_0^2 d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{16}{4}\,d\theta$

$= -4\theta \Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -4\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = -4\pi$

Flux $= \iint_R (M_x + N_y)\,dA = \iint_R (2xy + 1 - 2xy)\,dA$

~~$M_x = y + 1$~~

$M_x = 2xy + 1$

$N_y = -2xy$

$= \iint_R dA = \text{Area of } R$

$= \frac{1}{2}(\pi r^2)$

$= \frac{1}{2}(\pi \times 2^2) = 2\pi$

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Hence $f(x) = x^2yz + \frac{1}{2}\sin xz$ is a potential function of $F$ (corresponding to $C=0$)

Answer: $\int_{(0,1,1)}^{(1,2,1)} (2xyz + x\cos(xz))\,dx + (x^2z)\,dy + (x^2y)\,dz$

$= f(1,2,1) - f(0,1,1)$

$= 2 + \frac{1}{2}\sin 1 - (0 + \frac{1}{2}\sin 0)$

$= 2 + \frac{1}{2}\sin 1$

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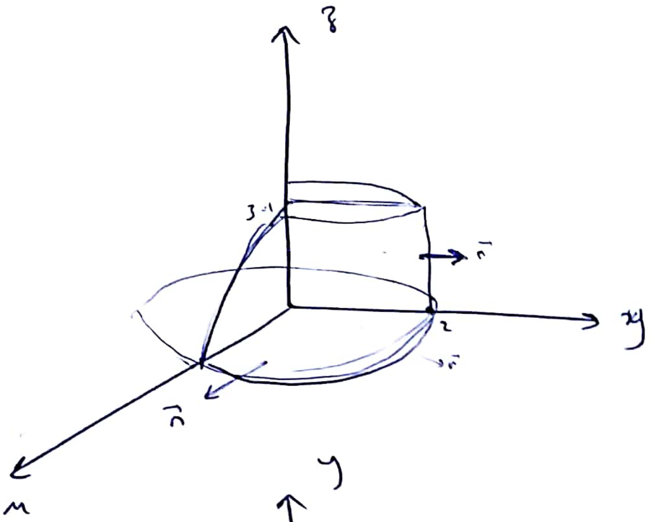
**Problem 3**

(20 pts) Let the surface $S$ be the portion of the cylinder $x^2+y^2=4$ in the **first octant** that lies **below** the plane $z=1$.

Find the outward flux of $\mathbf{F}=z\mathbf{i}+x\mathbf{j}+y\mathbf{k}$ across $S$. ("Outward" means that the normal vector points away from the $z$–axis.)




flux = $\iint_S F\cdot n\, d\sigma$

parametrize S: $x^2+y^2=4$:

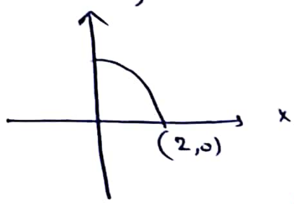
$\vec{r}\,(2\cos\theta, 2\sin\theta, r)$

$n\,d\sigma = \pm\, r_\theta \times r_r\, dA$

$r_\theta \times r_r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= \vec{i}\,(2\cos\theta, +2\sin\theta, 0)$

$n\,d\sigma = +(2\cos\theta, 2\sin\theta, 0)\, dr\, d\theta$

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so flux $= \int_0^{\frac{\pi}{2}}\int_0^{1} (r, 2\cos\theta, 2\sin\theta)\cdot(2\cos\theta, 2\sin\theta, 0)\, dr\, d\theta$

$= \int_0^{\frac{\pi}{2}}\int_0^{2} (2r\cos\theta + 4\sin\theta\cdot\cos\theta)\, dr\, d\theta$

$= \int_0^{\frac{\pi}{2}} \left[r^2\cos\theta + 4r\sin\theta\cdot\cos\theta\right]_0^2 d\theta = \int_0^{\frac{\pi}{2}} (4\cos\theta + 8\sin\theta\cdot\cos\theta)\, d\theta$

$8\cdot\frac{\sin 2\theta}{2} = 4\sin 2\theta$

$= \left[4\sin\theta - \frac{4\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}} = \left(4\sin\frac{\pi}{2} - 2\cos\pi\right) - (4\sin 0 - 2\cos 0)$

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$= 4+2+2$

$= 8$

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ZK

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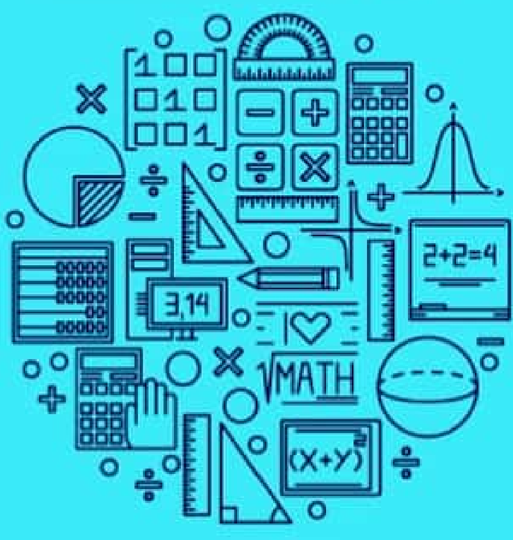


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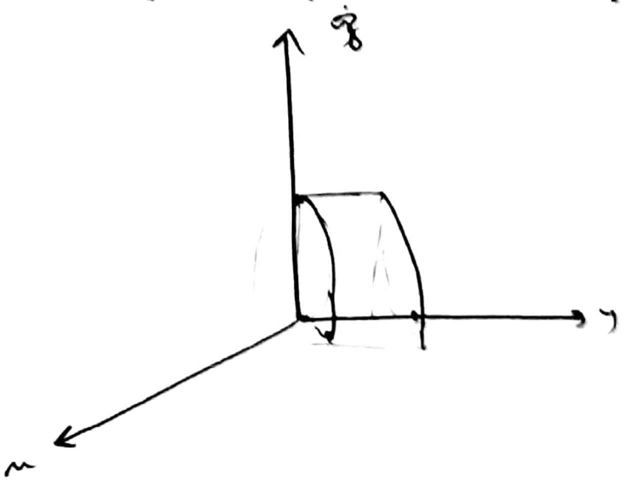
**Problem 4**

(20 pts) Integrate $G(x,y,z) = y\sqrt{x^2+4}$ over the surface cut from the parabolic cylinder $x^2+4z=4$ by the planes $y=0$, $y=1$, and $z=0$.

$x^2+4z=4$

$4z=4-x^2$

$z=1-\frac{1}{4}x^2$



$\iint_S G(x,y,z)\,d\sigma$

param S: $\vec{r}(x, y, 1-\frac{1}{4}x^2)$

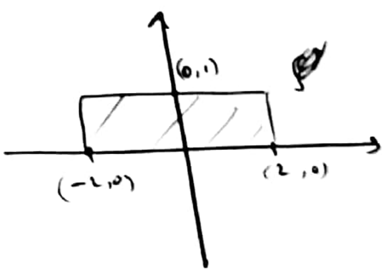
$d\sigma = |\vec{r}_x \times \vec{r}_y|\,dA$

$$r_x \times r_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\frac{1}{2}x \\ 0 & 1 & 0 \end{vmatrix}$$

$= (\frac{1}{2}x, 0, 1)$

$|r_x \times r_y| = \sqrt{(\frac{1}{2}x)^2+1} = \sqrt{\frac{1}{4}x^2+1}$

S':



So $\iint_S G(x,y,z)\,d\sigma$

$= \iint_{S'} y\sqrt{x^2+4} \cdot \sqrt{\frac{1}{4}x^2+1} \cdot dA$

$= \iint_{S'} \frac{1}{2}y\sqrt{x^2+4} \cdot \sqrt{x^2+4}\,dA = \iint_{S'} \frac{1}{2}y(x^2+4)\,dA = \int_{-2}^{2}\int_{0}^{1} \frac{1}{2}y(x^2+4)\,dy\,dx$

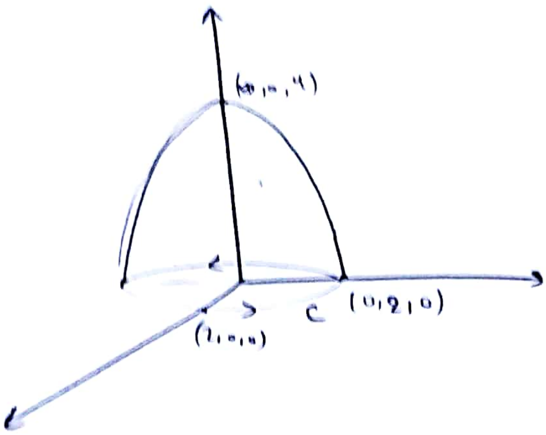
$= \int_{-2}^{2} \frac{y^2(x^2+4)}{4}\Big]_0^1\,dx = \int_{-2}^{2} \frac{(x^2+4)}{4}\,dx = \frac{1}{4}\left(\frac{x^3}{3}+4x\right)\Big]_{-2}^{2}$

$= \frac{1}{4}\left(\frac{8}{3}+8-\left(-\frac{8}{3}-8\right)\right) = \frac{1}{4}\left(2 \times \frac{32}{3}\right) = \frac{16}{3}$

**Problem 5**

Let $S$ be the portion of the paraboloid $z = 4 - x^2 - y^2$ lying above the $xy$-plane, and let $C$ be the circle $x^2 + y^2 = 4$ where $S$ meets the $xy$-plane. Find the counterclockwise (as viewed from above) circulation of the field $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ around $C$

(a)(8 pts) Directly




$$\text{Circulation} = \oint_C \vec{F}\cdot d\vec{r} = \oint \vec{F}\cdot\vec{T}\, ds$$

$$= \int_C M\,dx + N\,dy + P\,dz$$

parametrize $C: \vec{r}\,(2\cos t, 2\sin t, 0)$

$0 \le t \le 2\pi$

$$= \int_0^{2\pi} (xy, y, z)\Big|_{(2\cos t,\, 2\sin t,\, 0)} \cdot (-2\sin t, 2\cos t, 0)\, dt$$

$$= \int_0^{2\pi} (4\cos t \sin t, 2\sin t, 0)\cdot(-2\sin t, 2\cos t, 0)\, dt$$

$$= \int_0^{2\pi} (-8\cos t \sin^2 t + 4\sin t \cos t)\, dt$$

$$= \int_0^{2\pi} -8\sin^2 t\, d(\sin t) + \int_0^{2\pi} 2\sin 2t\, dt$$

$$= \left.\frac{-8\sin^3 t}{3}\right]_0^{2\pi} + \left(-\cos 2t\right)_0^{2\pi} = 0 + (-\cos 4\pi + \cos 0)$$

$$= 0 + 0 = 0$$

(b)(8 pts) By using Stokes' theorem over the surface $S$.

$$\text{Circulation} = \iint_S \vec{\nabla} \times \vec{F} \cdot n \, d\sigma$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y & z \end{vmatrix} = (0, 0, -x)$$

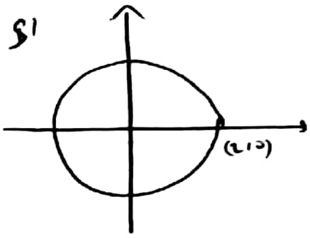
parametrize $S$: $\vec{r}(x, y, 4-x^2-y^2)$ where $z = f(x,y) = 4 - x^2 - y^2$

$$n \cdot d\sigma = + r_x \times r_y \cdot dA$$

$$= + (2x, 2y, 1) \, dA$$

so $\text{circulation} = \iint_{S'} (0, 0, -x) \cdot (2x, 2y, 1) \, dA$

S'




$$= \iint_{S'} -x \, dA = \int_0^{2\pi}\int_0^2 -r\cos\theta \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} -\frac{r^3}{3}\cos\theta \Big]_0^2 \, d\theta$$

$$= \int_0^{2\pi} -\frac{8}{3}\cos\theta \, d\theta$$

$$= -\frac{8}{3} \cdot \sin\theta \Big]_0^{2\pi}$$

$$= -\frac{8}{3} \cdot \sin 2\pi - \left(-\frac{8}{3}\sin 0\right)$$

$$= 0 - 0 = 0$$

(c)(8 pts) By using Green's theorem over an appropriate region in the xy-plane. (Hint: in the xy-plane we have $z=0$, so the vector field **F** no longer has a **k**-component.)

in xy-plane : $\vec{F}(xy, y)$

Using G.T over C:

$$\text{circulation} = \oint\iint_R (N_x - M_y)\,dA$$

R is Region bounded by C in xy-plane

$$= \iint_R (0 - x)\,dA$$

$N_x = 0$

$M_y = x$

$$= -\iint_R x\,dA$$

$$= -\int_0^{2\pi}\int_0^2 r\cos\theta \cdot r\,dr\,d\theta$$

$$= -\int_0^{2\pi}\int_0^2 r^2\cos\theta\,dr\,d\theta = -\int_0^{2\pi} \frac{r^3}{3}\cos\theta\Big]_0^2 d\theta$$

$$= -\int_0^{2\pi} \frac{8}{3}\cos\theta\,d\theta = -\frac{8}{3}\sin\theta\Big]_0^{2\pi}$$

$$= -\frac{8}{3}(\sin 2\pi - \sin 0) = -\frac{8}{3}(0)$$

$$= 0$$