Exercise 1. (10 points) Compute the area of the surface S defined by $z = 2\sqrt{x^2+y^2}, x \geq 0,$ $y \geq 0, 0 \leq z \leq 1.$

Exercise 2. Let S be the surface defined by $z = 1 - x^2 - y^2$, $z \geq 0$, and let C be the curve $x^2 + y^2 = 1$, $z = 0$ oriented counterclockwise as viewed from above. Consider the vector field $\mathbf{F} = yz\mathbf{i} + y\mathbf{j} - xy\mathbf{k}$. Compute the circulation $\oint_C \mathbf{F} \cdot dr = \oint_C \mathbf{F} \cdot T\, ds$ of F around C

(a) (8 points) directly by parametrizing the curve C

(b) (15 points) using Stokes' theorem.

6

Exercise 3. (22 points) Let S be the surface defined by $z = x^2 + y^2, 0 \leq z \leq 1$. Notice that S is open from the top. The surface S is oriented with a normal vector n that points away from the z-axis. Consider the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}$. Compute the flux integral $\iint_S \mathbf{F}.n\, d\sigma$ using the divergence theorem.

$z = r^2$, $z \leq 1$

5

Exercise 4. (15 points) Solve the IVP

$$\begin{cases} x^3 \dfrac{dy}{dx} - 2 + x^2 y = 0 \\ \\ y(1) = 1. \end{cases}$$

Exercise 5. (15 points) Change the following non exact DE to an exact DE and solve it

$$(8 - 7y + x^8 e^x)dx + x dy = 0.$$

11

Exercise 6. (15 points) Find all the real numbers a such that the solution of the following IVP

$$\begin{cases} \dfrac{dy}{dx} = 2xy^2 \\ \\ y(a) = -1 \end{cases}$$

is defined on $\mathbb{R}$.

ZK

MATH TUTORS !

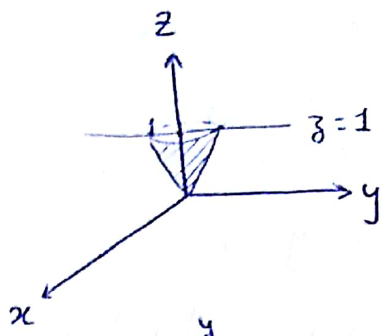
LET'S DO THINGS TOGETHER!

Contact Us

Razan Kachmar	Zeinab Zeitoun
70/ 892 234	70/ 748 800

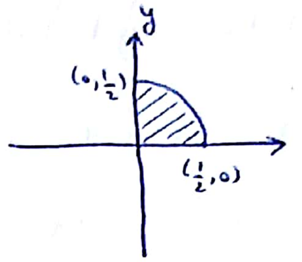
Exercise 1 :

$$S_A = \iint_R \frac{|\nabla f|}{|\nabla f \cdot k|}\, dA(x,y)$$



let $f = z - 2\sqrt{x^2+y^2}$

so $f(x,y,z) = z - 2\sqrt{x^2+y^2} = 0$ is the level surface



$z = 2\sqrt{x^2+y^2}$

$z = 1$

$1 = 2\sqrt{x^2+y^2}$

$x^2+y^2 = \frac{1}{4}$

$x \geq 0, y \geq 0$

$$\nabla f = (f_x, f_y, f_z) = \left(-\frac{2x}{\sqrt{x^2+y^2}}, \frac{-2y}{\sqrt{x^2+y^2}}, 1\right)$$

$\vec{k} = (0,0,1)$

$\nabla f \cdot k = 0 + 0 + 1$ so $|\nabla f \cdot k| = 1$

$$|\vec{\nabla} f| = \sqrt{\frac{4x^2}{x^2+y^2} + \frac{4y^2}{x^2+y^2} + 1} = \sqrt{\frac{5x^2+5y^2}{x^2+y^2}} = \sqrt{5}$$

$$S_A = \iint_R \frac{|\nabla f|}{|\nabla f \cdot k|}\, dA = \iint_R \sqrt{5}\, dA = \int_0^{\frac{\pi}{2}}\int_0^{1/2} \sqrt{5}\, r\,dr\,d\theta = \sqrt{5}\int_0^{\frac{\pi}{2}} \left.\frac{r^2}{2}\right]_0^{1/2} d\theta$$

$$= \frac{\sqrt{5}}{8}\, \theta\Big]_0^{\frac{\pi}{2}} = \frac{\pi\sqrt{5}}{16}$$

2nd Method:

$$S_A = \iint |r_u \times r_v|\, dA$$

let $u = x \quad v = y \quad z = 2\sqrt{u^2+v^2}$

$\vec{r} = u\vec{i} + v\vec{j} + 2\sqrt{u^2+v^2}\,\vec{k}$

$$\vec{r}_u = \left(1, 0, \frac{2u}{\sqrt{u^2+v^2}}\right) \qquad \vec{r}_v = \left(0, 1, \frac{2v}{\sqrt{u^2+v^2}}\right)$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{2u}{\sqrt{u^2+v^2}} \\ 0 & 1 & \frac{2v}{\sqrt{u^2+v^2}} \end{vmatrix} = \vec{i}\left(-\frac{2u}{\sqrt{u^2+v^2}}\right) - \vec{j}\left(\frac{2v}{\sqrt{u^2+v^2}}\right) + 1\vec{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\frac{4u^2}{u^2+v^2} + \frac{4v^2}{u^2+v^2} + 1} = \sqrt{\frac{5(u^2+v^2)}{u^2+v^2}} = \sqrt{5}$$

$$S_A = \iint \sqrt{5}\, dA = \text{(Similarly)}\ \frac{\pi\sqrt{5}}{16}$$

Exercise 2 :

(S) : $z = 1 - x^2 - y^2$, $z \geq 0$

(C) : $x^2 + y^2 = 1$ $\quad z = 0$ oriented counterclockwise

$F = yz\vec{i} + y\vec{j} + xy\vec{k}$

a) circulation $= \oint_C \vec{F}.\vec{T}\,ds = \int_a^b \vec{F}\big|_{r(t)} . \frac{\vec{V}(t)}{|\vec{V}(t)|} . |V(t)|\,dt = \int_a^b \vec{F}\big|_{r(t)} . \vec{V}(t)\,dt$

$x^2+y^2=1$ $\quad x = \cos t$, $y = \sin t$ $\quad$ so $\vec{r}(t) = (\cos t, \sin t, 0)$

$\vec{V}(t) = (\vec{r}(t))' = (-\sin t, \cos t, 0)$ $\quad 0 \leq t \leq 2\pi$

$\vec{F}\big|_{r(t)} = (0, \sin t, \cos t . \sin t)$

Circulation $= \int_0^{2\pi} (0, \sin t, \cos t.\sin t)(-\sin t, \cos t, 0)\,dt$

$= \int_0^{2\pi} (0 + \sin t \cos t + 0)\,dt = \int_0^{2\pi} \frac{\sin 2t}{2}\,dt = \left[\frac{-\cos 2t}{4}\right]_0^{2\pi} = -(0-0) = 0$

b) By stoke's theorem :

Circulation $= \iint_S \vec{n} . \operatorname{curl} F\,d\sigma$ $\quad$ where $\vec{n}\,d\sigma = \frac{\vec{r}_u \times \vec{r}_v}{\cancel{|r_u \times r_v|}} . \cancel{|r_u \times r_v|}\,dA$

$= \iint_R (\vec{r}_u \times \vec{r}_v) . (\operatorname{curl} F)\,dA$

$\vec{r} = (u, v, 1 - u^2 - v^2)$ (let $u = x$, $v = y$, so $z = 1 - u^2 + v^2$)

$\vec{r}_u = (1, 0, -2u)$

$\vec{r}_v\ (0, 1, -2v)$

$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ yz & y & xy \end{vmatrix} = \vec{i}\,(x) - \vec{j}\,(y - y) + (-z)\vec{k}$

$= (x, 0, -z)$

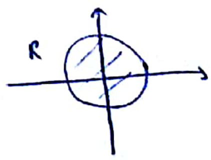
$\operatorname{Curl} F\big|_{\vec{r}} = (u, 0, u^2+v^2-1)$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = \vec{i}\,(2u) - \vec{j}\,(-2v) + \vec{k}\,(1)$

$= (2u, 2v, 1)$

so $\vec{n}\,d\sigma = +(2u, 2v, 1)\,dA$

R



Circulation $= \iint_R (2u, 2v, 1).(u, 0, u^2+v^2-1)\,dA$

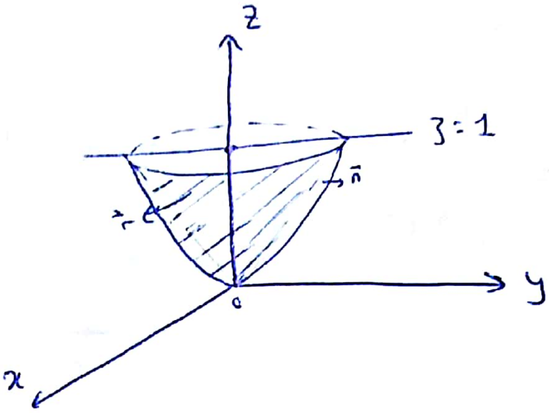
$= \iint_R (2u^2 + u^2 + v^2 - 1)\,dA \quad = \int_0^{2\pi}\int_0^1 (2r^2\cos^2\theta + r^2 - 1)\,r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 (2r^3\cos^2\theta + r^3 - r)\,dr\,d\theta = \int_0^{2\pi} \left[\frac{r^4}{2}\cos^2\theta + \frac{r^4}{4} - \frac{r^2}{2}\right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{2}\cos^2\theta - \frac{1}{4}$

$$= \int_0^{2\pi} \left[\frac{1}{2}\left(\frac{1}{2}+\frac{\cos 2\theta}{2}\right)-\frac{1}{4}\right]d\theta = \int_0^{2\pi}\frac{\cos 2\theta}{2}\,d\theta = \left[\frac{\sin 2\theta}{4}\right]_0^{2\pi} = 0-0=0$$

Exercise 3 : $F=(y,-x,z^2)$

(S) $z=x^2+y^2 \quad 0\le z\le 1$



$$\text{Flux} = \iiint_D \operatorname{div} F\, dV$$

$$\operatorname{div} F = \nabla\cdot F$$

$$= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(y,-x,z^2)$$

$$= 0+0+2z = 2z$$

D should be closed $\Rightarrow$ D is the paraboloid $z=x^2+y^2$ cut by $z=1$ & including its top : 

flux through paraboloid & its top :

$$\iiint_D 2z\, dV = \int_0^{2\pi}\int_0^1\int_0^1 2z\, r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left[z^2 r\right]_0^1 dr\,d\theta = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta$$

$=$ Area of circle $=\pi$

flux of $S=$ flux $D -$ flux $R \qquad R: x^2+y^2=1$

$$\text{Flux } R = \iint_R (M_x+N_y)\,dA = \iint_R 0+0\,dA = 0$$

Green's theorem

or $\text{Flux } R = \oint_C \vec{F}\cdot\vec{n}\,d\sigma$

$\vec{r}(t)=(\cos t,\sin t)$

since counterclockwise :

$\vec{r}(t)=\vec{r}(a+b-t)=\vec{r}(2\pi-t)$

$=(\cos(2\pi-t),\sin(2\pi-t))$

$=(\cos t,-\sin t)$

$\vec{v}(t)=(-\sin t,-\cos t)$

$\vec{F}|_{\vec{r}(t)}=(-\sin t,-\cos t)$

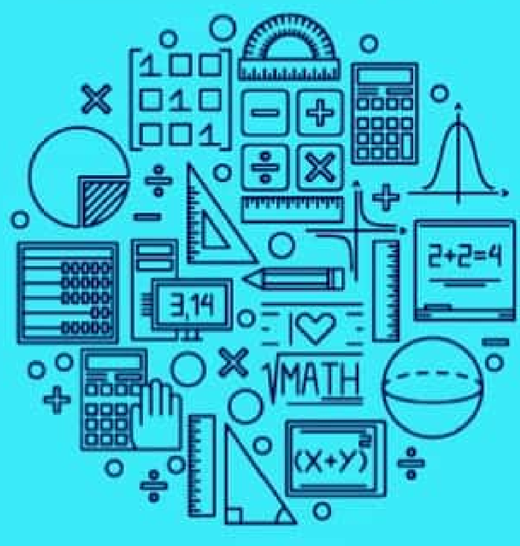
$$\text{flux } R = \int_0^{2\pi}\left(M\frac{dy}{dt}-N\frac{dx}{dt}\right)dt$$

$$=\int_0^{2\pi}\big((-\sin t)(-\cos t)-(-\cos t)(-\sin t)\big)dt = \int_0^{2\pi} 0\,dt = 0$$

$\therefore$ flux of $S = \pi-(0) = \pi$

ZK

MATH TUTORS !



LET'S DO THINGS TOGETHER!

Contact Us

Razan Kachmar
70/ 892 234

Zeinab Zeitoun
70/ 748 800

Exercise 4 :

$$x^3 \frac{dy}{dx} - 2x^2 y = 0$$

$$x^3 \frac{dy}{dx} + x^2 y = 2$$

$y' + \frac{1}{x} y = \frac{2}{x^3}$ Linear DE with $p(x) = \frac{1}{x}$

$\mu = e^{\int p(x)\,dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

DE × x

$xy' + y = \frac{2}{x^2}$ Integrate both sides with respect to x

$$\int (xy' + y)\,dx = \int \frac{2}{x^2}\,dx$$

$xy = \frac{-2}{x} + C$ $\quad y = -\frac{2}{x^2} + \frac{C}{x}$ $\quad y(1) = 1$

$1 = -2 + C$ $\quad \boxed{C = 3}$ $\quad \boxed{y = -\frac{2}{x^2} + \frac{3}{x}}$

Exercise 5 :

$$(8 - 7y + x^2 e^x)\,dx + x\,dy = 0$$

Change non Exact DE to an Exact DE

Multiply DE by μ

$$\underbrace{(8 - 7y + x^2 e^x)\mu}_{\tilde{M}}\,dx + \underbrace{x\mu}_{\tilde{N}}\,dy = 0$$

Want $\tilde{M}_y = \tilde{N}_x$

$\tilde{M}_y = 8\mu_y - 7\mu - 7y\mu_y + x^2 e^x \mu_y$ $\quad \tilde{N}_x = \mu + x\mu_x$

let $\mu = \mu(x)$ so $\mu_y = 0$ $\Rightarrow \tilde{M}_y = -7\mu$ $\quad \tilde{N}_x = \mu + x\mu_x$

$\tilde{M}_y = \tilde{N}_x$ $\quad -7\mu = \mu + x\mu_x$

$-8\mu = x\mu_x$ $\quad -8\mu = x\frac{d\mu}{dx}$

$-8\frac{dx}{x} = \frac{d\mu}{\mu} \Rightarrow -8\ln x = \ln \mu$

$$\boxed{\mu = \frac{1}{x^8}}$$

$\times \mu$:

$$\left(\frac{8}{x^8} - \frac{7y}{x^8} + e^x\right) dx + \frac{1}{x^7} dy = 0$$

(Check: $M_y = -\frac{7}{x^8}$, $N_x = -\frac{7}{x^8}$ $\Rightarrow$ Exact DE)

$\exists$ potential function $f(x,y)$ s.t $\nabla f = (f_x, f_y) = (M, N)$

$$f_x = M = \frac{8}{x^8} - \frac{7y}{x^8} + e^x \Rightarrow f = \int f_x\, dx = \int (8x^{-8} - 7y\, x^{-8} + e^x)\, dy$$

$$= \frac{8x^{-7}}{-7} - 7y\frac{x^{-7}}{-7} + e^x + g(y)$$

$$= (8+y)\frac{1}{x^7} + e^x + g(y)$$

$f_y = N$

$\frac{1}{x^7} + g_y(y) = \frac{1}{x^7}$ $\quad g_y(y) = 0 \quad g(y) = 0$ (take $c=0$)

So $f(x,y) = (8+y)\frac{1}{x^7} + e^x$

Solution $f(x,y) = C$

$$(8+y)\frac{1}{x^7} + e^x = C$$

$$\frac{8+y}{x^7} = C - e^x \qquad \boxed{y = (C - e^x)x^7 - 8}$$

Exercise 6:

$\frac{dy}{dx} = 2xy^2 \qquad \frac{dy}{y^2} = 2x\, dx \qquad -\frac{1}{y} = x^2 + c \qquad y = \frac{-1}{x^2+c}$

$y(a) = -1 \qquad -1 = \frac{-1}{x^2+a} \qquad x^2 + a = 1 \qquad x^2 = 1 - a$

$1 - a \geq 0$

$a \leq 1$