Chapter 2- 2020 Summer

A Set-theory Recap

Let A, B be two subsets of a set Ω

Union

$$A \cup B = \{\omega \in \Omega : \omega \in A \,\mathrm{or}\, \omega \in B\}$$

Intersection

$$A \cap B \equiv AB = \{\omega \in \Omega : \omega \in A \,\mathrm{and}\, \omega \in B\}$$

Complement

$$\bar{A} \equiv A^c = \{\omega \in \Omega : \omega \notin A\} = \Omega \setminus A$$

Difference

$$B \setminus A = B \cap \bar{A} = \{\omega \in \Omega : \omega \in B \,\mathrm{and}\, \omega \notin A\}$$

Notation
When A is a countable set, we denote with $|A|$ the number of points/elements in A (the size or cardinality of A)

A Set-theory Recap

Properties

Commutativity

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A$$

Associativity

$$(A \cup B) \cup C = A \cup (B \cup C), \qquad (A \cap B) \cap C = A \cap (B \cap C)$$

Distributivity

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Idempotency

$$A \cup A = A, \qquad A \cap A = A$$

De Morgan's laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}, \qquad \overline{A \cap B} = \bar{A} \cup \bar{B}$$

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \bar{A}_i, \qquad \overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \bar{A}_i$$

A Set-theory Recap. Poincaré Identities

Inclusion-Exclusion Principle (Poincaré Identity)

Let $A_i \subset \Omega$, $i = 1, \dots, n$

$$
\begin{aligned}
\left|\bigcup_{i=1}^{n} A_i\right| &= |A_1 \cup A_2 \cup \ldots \cup A_n| = \\
& \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\
& + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| \\
& - \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots A_n|
\end{aligned}
$$

For example, for $n = 3$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

A Set-theory Recap. Poincaré Identities

Using the De Morgan's law, it then follows

$$
\begin{aligned}
|\bigcap_{i=1}^{n} \bar{A}_i| &= |\Omega \setminus \bigcup_{i=1}^{n} A_i| \\
&= |\Omega| - |\bigcup_{i=1}^{n} A_i| \quad \text{since}\,\Omega \supseteq \bigcup_{i=1}^{n} A_i \\
&= |\Omega| - \sum_{i=1}^{n} |A_i| + \sum_{1\le i<j\le n} |A_i \cap A_j| \\
&\quad - \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| \\
&\quad + \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots A_n|
\end{aligned}
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An Example

In a group of 21 (Arabic-speaking) Lebanese, 16 speak French, 13 English, 4 Armenian, 9 English and French, 2 French and Armenian, 3 English and Armenian and 1 English, French and Armenian. How many speak only Arabic?

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Let A be the set of Armenophones, E the set of Anglophones, F the set of Francophones

The set of those who speak only Arabic is

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$$\bar{A} \cap \bar{E} \cap \bar{F} = \overline{A \cup E \cup F} = \Omega \setminus (A \cup E \cup F)$$

$$\begin{aligned} |\bar{A} \cap \bar{E} \cap \bar{F}| &= |\Omega| - |A \cup E \cup F| \\ &= |\Omega| - (|A| + |E| + |F|) + |A \cap E| + |A \cap F| + |E \cap F| \\ &\quad -|A \cap E \cap F| \\ &= 21 - (4 + 13 + 16) + 3 + 2 + 9 - 1 = 1 \end{aligned}$$

Sample Space and Events

The sample space Ω (or S) associated with an experiment is the set of all possible outcomes of such an experiment

$$\Omega = \{\omega_1, \ldots, \omega_n\}$$

A subset E of Ω, $E \subset \Omega$, is called an event
Informally, an event is a statement on the outcomes of a random experiment

We also assume that

- Ω is an event (the certain or universal event)
- if E is an event, so is its complement $\bar{E}$
  Hence, the empty set $\emptyset$ is an event (the impossible, or vacuous, event)
- the union of events is an event

Sample Space and Events

When E is just one outcome we say it is a simple event or a state (e.g. $E = \{\omega_3\}$)

When E is more than one, we say it is a composite event or, simply, event (e.g. $E = \{\omega_2, \omega_{21}\}$)

Sample Space

Describe the sample spaces for the following experiments

- roll a regular die

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$$\{1, 2, 3, 4, 5, 6\}$$

- toss n coins ($n = 4$)

Probability in a Simulation

Roll a die, what is the probability of $A =$"3 appears"?

The pseudo-code for the simulation to get the probability $P(A)$

Problem

How many ways can we roll three dice? In how many ways can three dice appear when they are rolled? How many possible numbers we get by rolling 3 dice (the order counts)?

Some problems

- How many ways are there for 2 persons to sit in 7 chairs that are in a row?

  $$7 \cdot 6 = (7)_2$$

  Sampling (the chairs) with ordering (it matters in a theater, say, where you sit) and without replacement (two people may not sit on the same chair)

- How many ways are there to make a 3-letter word (a string of 3 letters) using a 26-letter alphabet?

  $$26^3$$

- How many ways are there to form a 3-letter word if the letters must be different?

Sample Space

Describe the sample spaces for the following experiments

- roll a regular die

$$\{1, 2, 3, 4, 5, 6\}$$

- toss n coins $(n = 4)$

$$\{HHHH, HHHT, HHTH, HHTT, HTHH, HTTH, HTHT, HTTT, \\ TTTT, TTTH, TTHT, TTHH, THTT, THHT, THTH, THHH\}$$

- cast a regular die and, if 6 comes up, toss a coin

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$$6H, 6T, 1, 2, 3, 4, 5$$

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- cast a regular die and, if 6 comes up, toss a coin

$$6H, 6T, 1, 2, 3, 4, 5$$

- measure the time for the emission of radioactive particle from some atom

$$(0, \infty)$$

(this is a non-discrete case)

Events

Describe the events

1) when rolling a die, an even number comes up
2) when tossing n coins, the first $n-1$ outcomes are tails
3) measuring the time for the emission of radioactive particle from some atom, the emission occurs after 3 minutes

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1) $E = \{2, 4, 6\}$
2)
$$E = \{\overbrace{TTTTTTT}^{n-1}T, \overbrace{TTTTTTT}^{n-1}H\}$$
3) $E = (3, \infty)$ (using minutes as unit of time)

Sample Space and Events

Events are subsets of a set, the sample space Ω

Logical statements on events $\iff$ operation with sets

Both event A and event B occur (conjunction) $\iff$ $A \cap B$

at least one of A and B occurs (disjunction)/ either A or B or both occur $\iff$ $A \cup B$

the event A does not occur (negation) $\iff$ $\bar{A}$

the event A occurs but the event B does not $\iff$ $A \setminus B$

Some Terminology

If $A \cap B = \emptyset$, A and B are said to be disjoint or incompatible or mutually exclusive events
(that A and B both occur is impossible: the occurrence of one prevents the occurrence of the other)

If $A \subseteq B$, A is said to imply B (B occurs if A occurs)
In fact $\subseteq$ is the set-theoretic equivalent of $\Rightarrow$

Probability

The first two elements of a probabilistic model are the sample space and the notion of events.

The third element is the assignment of a probability to the events and outcomes of a random experiment

We need to formalize statements such as

the probability that when we roll a die an even number comes up is 1/2

Probability as Measure of Relative Frequencies

One interpretation views probability as a relative frequency (which can be justified a posteriori by the result known as the law of large numbers)

Carry out repeatedly and independently the same experiment a large number of times N (roll the same die in the same conditions N times)
record the number of times $S_N(E)$ the event E occurs ("an even number comes up")
assign to the event the probability $P(E) = S_N(E)/N$, N large, (the empirical limiting relative frequency in the N repetitions)

Probability as Measure of Relative Frequencies

The frequency definition of probability is based on the assumption that identical and independent experiments can be carried out (which is not always the case)

A priori there is no guarantee that the relative frequency should converge to a limit and if that is the case, it is not clear how large N should be for the approximation to be reliable

However even if there may be difficulties involved in defining probability in a mathematical form using repetitive events, this notion of probability is the basis of simulations, so you should keep it in mind

Probability in a Simulation

Roll a die, what is the probability of $A =$"3 appears"?

The pseudo-code for the simulation to get the probability $P(A)$

```
counter_a=0
for i from 1 to N
      roll a die
      if(die  shows 3)
          counter_a = counter_a+1
      end if
end for
return counter_a/N
```

Probability in a Simulation. Implementation

- Python

```
import numpy as np
np.random.seed(164)
N=1000000
die= np.arange(6)+1
a=np.random.choice(die, N, replace=True)
np.count_nonzero(a==3)/N
```

- R

```
set.seed(1364)
N=1000000
a=sample(1:6, replace=T, N)
length(which(a==3))/N
```

- Matlab

```
rng(6356)
N=1000000;
a = randsample(6, N, true);
sum(a==3)/N
```

Probability

In general, the assignment of a probability to an event is rather subtle

Sometimes there are some natural choices, for example based on the existence of symmetries in the random experiment at hand
Sometimes the choice will be subjective (probability assignments may differ from individual to individual)

Let us now consider a mathematical (axiomatic) definition of probability:
probability as a function that satisfies some properties (a version of these properties are indeed verified by the relative frequency)

Discrete Probability Space

Let the sample space Ω be non-empty and countable for the rest of this chapter

Third element necessary to complete the description of a probabilistic model is a function

$$P : \text{Set of Events} \longrightarrow [0, 1]$$

called probability that verifies the following two axioms

A1) $P(\Omega) = 1$

The certain event has probability 1

A2) σ-additivity (countable additivity)

For every sequence $(A_i)_{i \in \mathbb{N}}$ of disjoint/mutually exclusive events, $A_i \cap A_j = \emptyset$, $i \neq j$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

(Ω, P) is called a *discrete probability space*, with Ω its sample space, and subsets of Ω the events

Probability. Properties

For a discrete probability space (Ω, P), the following statements hold true (all follow from A1 and A2)

S1) The impossible event has probability zero

$$P(\emptyset) = 0$$

Proof. If $A_i = \emptyset$, $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} A_i = \emptyset$

$$P(\emptyset) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} P(\emptyset)$$

which holds iff $P(\emptyset) = 0$. $\square$

Probability. Properties

S2) Finite additivity

Let A_i, $i = 1, \ldots, n$, be a finite family of disjoint events, $A_i \cap A_j = \emptyset$, $i \neq j$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

This follows from countable additivity, A2, setting $A_k = \emptyset$ for all $k \geq n$. Thus it is a weaker notion than countable additivity.

When Ω is finite, we can equally define the probability space using axioms A1 and A2 or A1 and S2 (finite additivity)

Probability. Properties

Other consequences of A1 and A2 (draw the corresponding Venn diagram if in doubt)

S3) $P(\bar{A}) = 1 - P(A)$

$$1 = P(\Omega) = P(A \cup \bar{A}) = P(A) + P(\bar{A})$$

since $A \cap \bar{A} = \emptyset$ $\square$

S4) For any $A, B \subseteq \Omega$

$$P(B \setminus A) = P(B) - P(A \cap B)$$

Since $B = (B \setminus A) \cup (A \cap B)$, with $B \setminus A$ and $(A \cap B)$ disjoint, then

$$P(B) = P(B \setminus A) + P(A \cap B)$$

hence the result $\square$

Probability. Properties

S5) For any or any $A, B \subseteq \Omega$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Since $A \cup B = A \cup (B \setminus A)$, and A and $B \setminus A$ are disjoint,

$$P(A \cup B) = P(A) + P(B \setminus A) = P(A) + P(B) - P(A \cap B)$$

using S4 $\square$

Inclusion-Exclusion Formulae (Poincaré's Identities)

(Ω, P) a discrete probability space.
For any $n \geq 1$ and for any choice of sets (events) $A_1, \ldots A_n \subseteq \Omega$

$$P(A_1 \cup \ldots \cup A_n) = \sum_{k=1}^{n} (-)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} P(A_{i_1} \cap \ldots \cap A_{i_k})$$

$$P(A_1 \cap \ldots \cap A_n) = \sum_{k=1}^{n} (-)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} P(A_{i_1} \cup \ldots \cup A_{i_k})$$

They can be proven by induction
They are actually valid on any probability space (finite, countable or uncountable)

Example

For example for $n = 4$, calling the events $A_1 = A$, $A_2 = B$, $A_3 = C$, $A_4 = D$

$$\begin{aligned} P(A \cup B \cup C \cup D) &= P(A) + P(B) + P(C) + P(D) \\ &\quad - P(A \cap B) - P(A \cap C) - P(A \cap D) \\ &\quad - P(B \cap C) - P(B \cap D) - P(C \cap D) \\ &\quad + P(A \cap B \cap C) + P(A \cap B \cap D) \\ &\quad + P(A \cap C \cap D) + P(B \cap C \cap D) \\ &\quad - P(A \cap B \cap C \cap D) \end{aligned}$$

Alternating sum of the probabilities of each event (4 terms), each possible pair of events (6), each possible triple of events (4), each possible quadruple (1)

When to use the Poincaré formula

Very often if one has to compute the probability that at least one event occurs or the probability that no event occurs, the Poincaré formula is most useful

Indeed, let A_i be $i = 1, \ldots, n$ n events

$P(A_1 \cup \ldots \cup A_n)$ is the probability that *at least* one of the n events occurs

$P(\bar{A}_1 \cap \ldots \cap \bar{A}_n)$ is the probability that none occurs

By the Poincaré formulae, these probabilities can be written in terms of probability involving fewer events, which are often easier to compute

Probability of events and outcomes

(Ω, P) discrete probability space, $A = \{\omega_{i_1}, \ldots, \omega_{i_k}\}$ a compound event, then its probability is

$$P(A) = \sum_{j=1}^{k} P(\omega_{i_j})$$

with the restriction $1 = P(\Omega) = \sum_{\omega \in \Omega} P(\omega)$
Example

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\} \qquad P(\omega_i) = p, \forall i$$
$$A_1 = \{\omega_1, \omega_2\}, A_2 = \{\omega_1, \omega_3\}, A_3 = \{\omega_1, \omega_4\}$$

Find $P(A_i)$, $P(A_i \cap A_j)$, $P(A_1 \cap A_2 \cap A_3)$, $P(A_i \cup A_j)$ etc.

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Example

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\} \qquad P(\omega_i) = p, \forall i$$
$$A_1 = \{\omega_1, \omega_2\}, A_2 = \{\omega_1, \omega_3\}, A_3 = \{\omega_1, \omega_4\}$$

Find $P(A_i)$, $P(A_i \cap A_j)$, $P(A_1 \cap A_2 \cap A_3)$, $P(A_i \cup A_j)$ etc.

$P(\omega_i) = 1/4$, $P(A_i) = 2/4$, $P(A_i \cap A_j) = 1/4 = P(A_1 \cap A_2 \cap A_3)$
$P(A_i \cup A_j) = 3/4$, $P(A_1 \cup A_2 \cup A_3) = P(\Omega) = 1$

Uniform Probability Space

A discrete probability space (Ω, P) which is finite (that is, Ω is finite) and such that the outcomes $\omega_i \in \Omega$ are equiprobable is called a *uniform probability space*.
In this case, then

$$P(\omega_i) = 1/|\Omega|$$

The previous page example is such a space
More generally for any event E (a subset of Ω)

$$P(E) = \frac{|E|}{|\Omega|}$$

The probability of an event is the ratio of the number of cases that are favorable to it, to the number of possible cases, when there is nothing to make us believe that one case should occur rather than any other [Laplace]

Thus the problem of computing a probability of an event becomes the problem of counting the number of its elements

Equiprobability

It is necessary that all points in the sample space be equiprobable for computing probability via simple counting $P(E) = |E|/|\Omega|$

Suppose we want to compute the probability of getting a three by summing the numbers that turn up on tossing two dice

$$E = \text{"the sum of two throws is 3"}$$

Since the sum is symmetric, we can think of using the following sample space, where $[i,j]$ is the unordered couple, with i the number for one of the dice and j for the other

$$\tilde{\Omega} = \begin{pmatrix} [1,1] & [1,2] & [1,3] & [1,4] & [1,5] & [1,6] \\ & [2,2] & [2,3] & [2,4] & [2,5] & [2,6] \\ & & [3,3] & [3,4] & [3,5] & [3,6] \\ & & & [4,4] & [4,5] & [4,6] \\ & & & & [5,5] & [5,6] \\ & & & & & [6,6] \end{pmatrix}$$

Equiprobability

While there may be nothing wrong in using $\tilde{\Omega}$, it should be noticed however its simple events are not equally probable For example,

$$p([1,1]) = 1/36$$
$$p([1,2]) = p((1,2) \cup (2,1)) = p((1,2)) + p((2,1)) = 2/36,$$

$[i,j]$ unordered couple, (i,j) ordered couple
$[i,j]$ one die shows the i-th face the other the j-th face
(i,j) the first die shows i the second j, if we throw them one after the other or, if you toss them at the same time, just color the dice differently: the red die shows i, the blue die j

$$[i,j] = \{(i,j),(j,i)\}$$

Thus we may not compute the probability by simple counting the number of the points in the space that are favorable to the event (one point $[1,2]$) and divide by the size of the space 21 We would get $1/21$, instead of the correct probability which is $2/36$

Equiprobability

Instead we can use the following equiprobable sample space

$$\Omega = \begin{pmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{pmatrix}$$

whose $|\Omega| = 36$ outcomes are all equiprobable $P((i,j)) = 1/36$, thus $|E = \{i + j = 3\}| = 2$, $P(E) = |E|/|\Omega| = 2/36$

More generally (exercise) the probability of rolling a sum of k, with two dice is

$$P(\{i + j = k\}) = \frac{6 - |7 - k|}{36}, \quad k = 2, \ldots, 12$$

Combinatorics. Multiplication Rule

First rule of counting. Multiplication Rule

If an object is formed by making a succession of choices such that there are n_1 possibilities for the first choice, n_2 possibilities for the second (after the first choice is made) etc. then the total number of objects that can be made by making a set of choices is

$$|E| = n_1 \cdot n_2 \cdots$$

For the rule to apply, the *number* of available possibilities of each choice must be the same irrespective of which choice was made previously (n_i for the i-th choice, which may be different from n_j). However the *set* of available possibilities may differ and depend on the choice made at the previous stages

Book Problem 31

Beethoven wrote 9 symphonies, 5 piano concertos, and 32 piano sonatas.

a) How many ways are there to play first a Beethoven symphony and then a Beethoven piano concerto?

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$$9 \cdot 5 = 45$$

You may find it helpful to draw a tree

b) The manager of a radio station decides that on each successive evening (7 days per week), a Beethoven symphony will be played followed by a Beethoven piano concerto followed by a Beethoven piano sonata. For how many years could this policy be continued before exactly the same program would have to be repeated?

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$$9 \cdot 5 \cdot 32 = 1440 \, \text{days} \ \approx 4 \, \text{years}$$

Problem

How many ways can we roll three dice? In how many ways can three dice appear when they are rolled? How many possible numbers we get by rolling 3 dice (the order counts)?

We have a succession of three choices (one per die) and each die represents a multiple choice of six possibilities $n_i = 6$, thus there are

$$6 \cdot 6 \cdot 6 = 216$$

possible rolls

This is obvious if you think of tossing the dice one after the other rather than simultaneously (although it does not matter) The set of choices at each throw of the dice in this case does not change based on what happens at the previous stage (it is always one of the numbers $\{1, 2, 3, 4, 5, 6\}$ (and thus the number does not change which is only thing that matters for the multiplicative rule)

Problem

In how many ways can two dice show different faces?

Problem

In how many ways can two dice show different faces?

$$6 \cdot 5$$

In this case there are 6 available possibilities for the first choice. Five different possibilities for the second die. Which five depends on the first choice (e.g. if 3 shows up on the first roll, the set of available possibilities is $\{1, 2, 4, 5, 6\}$, if 2 shows up, such set is $\{1, 3, 4, 5, 6\}$), but there are always 5 possibilities for the second die, so the first rule of counting as formulated above still applies

Problem

How many different calendars are possible for a year?

Problem

How many different calendars are possible for a year?

$$7 \cdot 2 = 14$$

Each year starts on one of the seven days (Sunday, Monday, ..., Saturday). Each year is either a leap year (i.e., it includes February 29) or not

Standard Ways of Counting and Basic Probabilistic Models

We are going to see now two standard ways of counting which can be used in the majority of combinatorial problems

- Sampling Methods
- Allocation Methods

Basic Probabilistic Models. Sampling Model

Consider a population of n individuals (people, cards, numbered balls), *i.e.* an aggregate of *n distinguishable* elements without regard to their order (think of an urn containing nF numbered balls)

Choose an individual from the population (a sample of size 1) (that is, draw one ball from the urn)
How many ways can we do this?

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How many ways can we do this?
The answer is n

Choose k individuals successively, one at a time
How many ways can we do this?

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It depends

Basic Probabilistic Models. Sampling Model

It depends on whether or not a chosen individual is returned to the population before another is chosen (that is, on whether or not we put the ball we have drawn back into the urn)

It also depends on whether the sample of size k is ordered or not (namely, ordering means that the same individuals chosen in a different order are considered to be a different sample- drawing 1 before 2 is considered different from drawing 2 before 1)

The two kinds of sampling models are called

sampling with replacement
we put the drawn ball back into the urn or we replace the ball we have drawn in the urn

sampling without replacement
we do not replace the drawn balls

In addition we need to consider the sample to be **ordered** or **un-ordered**

Basic Probabilistic Models

Thus we will consider the following $2 \cdot 2 = 4$ cases

- Sampling with replacement and with ordering
- Sampling without replacement and with ordering
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Sampling with replacement and with ordering

I. Sampling with replacement and with ordering

The first individual is from a population with n individuals

$$n_1 = n$$

We return the individual to the population, thus the set (and hence the number) of possibilities for the choice of a second individual is the same as for the first

$$n_2 = n$$

Thus repeating the argument, using the multiplication rule (first rule), we find that there are

$$n_1 \cdots n_k = n \cdots n = n^k$$

ways to draw an (ordered) sample of size k from a population of n individuals

Sampling with replacement and with ordering

n^k counts the number of different ordered k-tuple from a population of size n

Ordering means, *e.g.*, that $(1, 2, 1, 3)$ is considered different from $(1, 2, 3, 1)$, $(2, 1, 3, 1)$, etc.

Put it differently, the sample space Ω is the set of ordered k-tuples

$$\Omega = \{(a_1, a_2, \ldots, a_k),\ a_i = 1, ..., n\}$$

Let us write down explicitly the number of ordered samples of size $k = 2$ that can be obtained from $n = 3$ objects, if an object may appear more than once ("sampling with replacement")

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$$\begin{array}{ccc} (1,1), & (1,2), & (1,3) \\ (2,1), & (2,2), & (2,3) \\ (3,1), & (3,2), & (3,3) \end{array}$$

Sampling without replacement and with ordering

II. Sampling without replacement and with ordering

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The first individual is chosen from a population with n individuals

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We do not return the individual to the population (we do not replace it), so now the population contains $n-1$ individuals. The number of possibilities for the choice of a second individual is

$$n_2 = n - 1$$

Thus using the multiplication rule, there are

$$n \cdot (n-1) \cdots (n-k+1) = (n)_k \equiv \frac{n!}{(n-k)!} \equiv P_{k,n}$$

ways of drawing an ordered sample of size k from a population of n individuals without replacement

$(n)_k$ is called lower or falling factorial

Sampling without replacement and with ordering

Notice that in this case the k choices are not independent, as the earlier individuals we drew affect the *set* of possibilities of later individuals, but the *number* of possibilities of a *given* later draw of an individual is not affected by earlier draws, so the first rule still applies

Sampling without replacement and with ordering

In this case (sampling without replacement and with ordering) the sample space is the set of ordered k-tuples with the constraint that no two elements can be the same (known also as k-permutations or ordered k-sets)

$$\Omega = \{(a_1, a_2, \ldots, a_k),\ a_i = 1, \ldots, n,\ \ a_i \neq a_j \,\mathrm{if}\, i \neq j\}$$

Let us write down explicitly the number of ordered pairs ($k = 2$) that can be obtained from a populations of $n = 3$ objects, if an object may not appear more than once ("sampling without replacement")

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$$\begin{array}{cc} (1,2) & (1,3) \\ (2,1) & (2,3) \\ (3,1) & (3,2) \end{array}$$

Some problems

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  Sampling (the chairs) with ordering (it matters in a theater, say, where you sit) and without replacement (two people may not sit on the same chair)

- How many ways are there to make a 3-letter word (a string of 3 letters) using a 26-letter alphabet?

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  $$26^3$$

- How many ways are there to form a 3-letter word if the letters must be different?

  $$26 \cdot 25 \cdot 24 = (26)_3$$

Sampling with ordering

Drawing an ordered sample of size k from an urn containing n balls a_i label of the ball selected at i-th step $a_i = 1, \ldots, n$		
Type	sample space Ω $(a_1, a_2, \ldots, a_k)$ ordered k-tuple	$\lvert\Omega\rvert$
with replacement	$\Omega = \{(a_1, a_2, \ldots, a_k) : a_i = 1, \ldots, n\}$	n^k
without replacement $n \geq k$	$\Omega = \{(a_1, a_2, \ldots, a_k) : a_i = 1, \ldots, n, \; a_i \neq a_j \text{ if } i \neq j\}$ (set of k-permutations of an n-set)	$n(n-1)\cdots(n-k+1)$ $= \frac{n!}{(n-k)!} \equiv (n)_k \equiv P_{k,n}$

For example, if $n = 3$, $k = 2$, the detailed sample spaces are:

with replacement	without replacement
(1, 1), (1, 2), (1, 3) (2, 1), (2, 2), (2, 3) (3, 1), (3, 2), (3, 3)	(1, 2), (1, 3) (2, 1), (2, 3) (3, 1), (3, 2)

Permutations

As we have seen $(n)_k \equiv P_{k,n} \equiv n!/(n-k)!$ denotes the number of ordered samples of size k drawn from a population of n individuals without replacement

A sample of size n (i.e. $k = n$) includes therefore the whole population and represents a re-ordering or *a shuffling* of its elements, generally referred to as an n-permutation or simply a permutation

The number of different permutations of n elements is then

$$(n)_n = P_{n,n} = n!$$

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- How many different words (i.e. strings of letters) can be obtained from the letters "BAALBEK" (using all letters)?

$$\frac{7!}{2!2!}$$

shuffling the two A's and the two B's does not change the word

Sampling without Ordering

We now consider the case in which the order of the elements in the sample is irrelevant/disregarded

In this case too we have to distinguish between sampling with or without replacement

A word on terminology: we will always explicitly specify whether the elements in the samples are ordered or not, unless it is clear from the context. Sometimes however the word "sample" is used to refer to what we have called "an ordered sample" and the word "population" is used to refer to an aggregate of elements without regard to their order. We have indeed used population in this sense, but when we consider sampling from a population, we will mostly talk about "unordered samples of size k" instead of sub-populations of size k. In some books, you may find that the problem of counting the number of possible unordered samples of size k from a population of size n is referred to as counting the number of subpopulations of size k of a given population of size n

Sampling without replacement and without ordering

III. Sampling without replacement and without ordering
Let's start by constructing explicitly the sample space of unordered k-tuple sampled *without replacement*

$n = 4$, $k = 2$, $\{1, 2, 3, 4\}$ the individuals in the population

The possible unordered samples of size $k = 2$ obtained without replacement are

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The possible unordered samples of size $k = 2$ obtained without replacement are

$$\Omega = \{[1, 2], [1, 3], [1, 4], [2, 3], [2, 4], [3, 4]\}$$

because we do not count as different the samples in which the individual 1 has been drawn before the individual 2 or after it

Notice however that while the order of the sampling is irrelevant, different individuals make a different sample (the individuals are distinguishable/labelled/numbered) so $[1, 2] = [2, 1] \neq [1, 3]$

Sampling without replacement and without ordering

For general n and k

$$\Omega = \{[a_1, \ldots, a_k] : a_i = 1, \ldots, n,\ a_i \neq a_j, i \neq j\}$$

where $[a_1, \ldots, a_k]$ indicates an unordered k-tuple whose elements are $a_1, \ldots, a_k$

Since all the k individuals in a sample of size k are different (no replacement) and the sample is unordered, we can actually consider selecting all the individuals together (that is, drawing k numbered balls all together or one at a time is the same thing)

Sampling without replacement and without ordering

How many elements are in Ω?

We can sample one individual at a time, or equivalently, draw k individuals from the population

An ordered k-tuple from the population can be obtained from an unordered one by numbering its elements

There are $k!$ different ways of numbering k elements

Thus there are exactly $k!$ times as many ordered samples of size k as there are unordered samples of size k

Indeed we can obtain any ordered sample of size k by shuffling one specific sample of size k (unordered sample) and since the elements in the sample are not repeated (sampling without replacement) there are $k!$ ordered samples for each unordered one (think of the "BEIRUT" example above, where $k = 6$)

Sampling without replacement and without ordering

Hence, the number of unordered samples of size k from a population of size n when the sampling is carried out without replacement is

$$\frac{P_{k,n}}{k!} = \frac{(n)_k}{k!} = \frac{n!}{(n-k)!k!} \equiv \binom{n}{k} \equiv C_{k,n}$$

the binomial coefficient

Some properties of the binomial coefficient

$$\binom{n}{k} = \binom{n}{n-k}, \quad \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$$

Exercises

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$$\binom{30}{3}3! = (30)_3 = 30 \cdot 29 \cdot 28 = P_{3,30}$$

choose three people, $\binom{30}{3}$, and then consider all possible orderings (3!)