Math 202 - Quiz 1 (Fall 17)

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he answer to question 3 on its sheet. The first two questions have extra sheets for you to write your answers on them. Any part of your answers written on the wrong sheet will not be graded. A sheet of paper has two sides, you can write on both of them.

- There are 3 problems in total. Some questions have several parts to them. Make sure that you attempt them all.
- This is a closed book exam and no calculators are allowed.

Name :

ID # :

Section : 4

Q1	13+13+13+13
Q2	19+14
Q3	0
TOTAL	85

Problem 1

(13 points each) Solve the following IVPs:

i-

$$y' = -\frac{y}{x} + \cos(x) \quad ; \quad y(1) = 4$$

ii-

$$y' = -\frac{2xe^{y^2} + 2xy\cos(x^2)}{2x^2ye^{y^2} + \sin(x^2)} \quad ; \quad y(0) = 0$$

iii-

$$y' = \frac{y}{x} + \frac{1}{\cos(\frac{y}{x})} \quad ; \quad y(1) = \frac{\pi}{2}$$

iv-

$$y' = -y^2e^x \quad ; \quad y(0) = -1$$

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i - $y' = -\frac{1}{x}y + \cos x \qquad y(1) = 4$

Linear eqtn $\quad y' = P(x)y + Q(x)$

$$y = y_0 e^{\int_{x_0}^{x} P(u)du} + \int_{x_0}^{x} Q(v) e^{\int_v^x P(u)du} dv$$

$$y = 4e^{\int_1^x -\frac{1}{u} du} + \int_1^x \cos v \, e^{\int_v^x -\frac{1}{u} du} dv$$

$$4e^{-\ln u|_1^x} = 4e^{-\ln x} = \frac{4}{x}$$

$$y = \frac{4}{x} + \int_1^x \cos v \, e^{-\ln u|_v^x} dv$$

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$$y = \frac{4}{x} + \int_1^x \cos v . e^{-\ln x} . e^{\ln v} dv$$

$$y = \frac{4}{x} + \int_1^x \cos v . x^{-1} . v \, dv$$

$$y = \frac{4}{x} + \frac{1}{x}\int_1^x v\cos v \, dv$$

$\int fg' = fg - \int f'g$

$$y = \frac{4}{x} + \frac{1}{x}(v\sin v + \cos v|_1^x)$$

$\int v\cos v = v\sin v - \int \sin v$
$= v\sin v + \cos v$

$$y = \frac{4}{x} + \frac{1}{x}(x\sin x + \cos x - \sin 1 - \cos 1)$$

$$y = \frac{4}{x} + \frac{1}{x}(x\sin x + \cos x - 1)$$

$$y = \frac{4}{x} + \sin x + \frac{\cos x}{x} - \frac{1}{x}$$

$$y = \frac{3}{x} + \frac{\cos x}{x} + \sin x$$

ii- $\bullet\, h = 2xe^{y^2} + 2xy\cos(x^2)$

$\bullet\, k = 2x^2ye^{y^2} + \sin(x^2)$

$\bullet\, h_y = 2x \cdot 2ye^{y^2} + 2x\cos(x^2) = 4xye^{y^2} + 2x\cos(x^2)$

$\bullet\, k_x = 4xye^{y^2} + 2x\cos(x^2)$

~~non not exact!~~

$$\cancel{\frac{h_y - k_x}{k} = \frac{4xye^{y^2} + 2x\cos(x^2) - 4xye^{y^2} + 2x\sin(x)^2}{2x^2ye^{y^2}}}$$

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$h_y = k_x$ $\therefore$ exact

$$0 = \int_0^x 2u e^{0^2} + 2u(0)\cos(u^2)\,du + \int_0^y 2x^2 v e^{v^2} + \sin(x^2)\,dv$$

$$\int_0^x 2u\,du + \int_0^y 2x^2 ve^{v^2} + \sin(x^2)\,dv = 0$$

$$u^2\Big|_0^x + x^2e^{v^2} + v\sin(x^2)\Big|_0^y = 0$$

$$x^2 + x^2e^{y^2} + y\sin(x^2) - x^2e^0 + 0 = 0$$

$$\cancel{x^2} + x^2e^{y^2} + y\sin(x^2) - \cancel{x^2} = 0$$

$$x^2e^{y^2} + y\sin(x^2) = 0$$

$c = v^2$

$\frac{dc}{dv} = 2v$

$$\int ve^{v^2} = \int \frac{ve^c}{2v}\,dc$$

$$= \int \frac{1}{2}e^c\,dc$$

$$= \frac{1}{2}e^c$$

$$= \frac{1}{2}e^{v^2}$$

iii - sub 't' in $f(tx, ty)$

$$y' = \frac{ty}{tx} + \frac{1}{\cos\left(\frac{ty}{tx}\right)} = \frac{y}{x} + \frac{1}{\cos\left(\frac{y}{x}\right)}$$

$\therefore$ homogenous

so $z = \frac{y}{x}$

$y = zx$

$y' = z + xz'$

$z(1) = \frac{\frac{\pi}{2}}{1} = \frac{\pi}{2}$

$$z + xz' = z + \frac{1}{\cos z}$$

$$xz' = \frac{1}{\cos z}$$

$$z' = \frac{1}{x\cos z} = \frac{1}{x} \cdot \frac{1}{\cos z}$$

seperable equation

$$\cancel{\int_1^x \frac{1}{u}\,du = \int_{\frac{\pi}{2}}^{z} \frac{1}{\cos v}\,dv = \sec v\,dv}$$

$$\cancel{\ln u\Big|_1^x = \ln|\cos v|\Big|_{\frac{\pi}{2}}^{z} \quad \sec v \tan v\Big|_{\frac{\pi}{2}}^{z}}$$

$$\cancel{\ln x - 0 = \ln(\cos z) - \ln\left(\cos\frac{\pi}{2}\right)}$$

$$\cancel{\ln x = \ln(\cos z) - \ln 0}$$

$$\cancel{\ln x = \sec z \tan z - \sec\frac{\pi}{2}\tan\frac{\pi}{2}}$$

$$\int_1^x \frac{1}{u}\,du = \int_{\frac{\pi}{2}}^{z} \cos v\,dv$$

$$\ln x - \ln 1 = \sin z - \sin\frac{\pi}{2}$$

$$\ln x = \sin z - 1$$

$$\sin z = \ln x + 1$$

$$\sin\frac{y}{x} = \ln x + 1$$

$$\frac{y}{x} = \sin^{-1}(\ln x + 1)$$

$$y = \cancel{x}\ \cancel{\operatorname{cosec}(\ln x+1)}$$

$$y = x \operatorname{cosec}(\ln x + 1)$$

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~~iii~~

iv - $y' = -y^2 e^x$ $\qquad y(0) = -1$

$$y' = -e^x y^2$$

seperable equation

v^{-2}

$-v^{-1}$

$$\int_{-1}^{y} \frac{1}{v^2}\,dv = \int_0^x -e^u\,du$$

$$-\frac{1}{v}\Big|_{-1}^{y} = -e^u\Big|_0^x$$

$$-\frac{1}{y} - \left(-\frac{1}{-1}\right) = -e^x + e^0$$

$$-\frac{1}{y} - 1 = -e^x + 1$$

$$+\frac{1}{y} = 2 - e^x$$

$$+\frac{1}{2-e^x} = y$$

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Problem 2

(19 points each) Solve the following IVPs:

i-

$$y''' - 3y'' + 2y' = 4x \quad ; \quad y(0) = 0 \quad , \quad y'(0) = 3, \quad , \quad y''(0) = 4$$

ii-

$$x^2y'' - xy' = x^3e^x \quad ; \quad y(1) = 0 \quad , \quad y'(1) = 3e$$

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i- $y_c = \quad m^3 - 3m^2 + 2m = 0$ $\qquad y = y_c + y_p$

$m(m^2 - 3m + 2) = 0$

$m(m^2 - 2m - m + 2) = 0$

$m(m(m-2) - (m-2)) = 0$

$m(m-1)(m-2) = 0$

$m = 0 \quad m = 1 \quad m = 2$

$FS: e^{0x}, e^{1x}, e^{2x}$

$y_c = c_1 + c_2e^x + c_3e^{2x}$

$y_p = Ax + B$ x

• $y_p = Ax^2 + Bx$

$y'_p = 2Ax + B$

$y''_p = 2A$

$y'''_p = 0$

$0 - 6A + 4Ax + 2B = 4x + 0$

$4Ax - 6A + 2B = 4x + 0$

$4A = 4 \qquad 2B - 6A = 0$

$\underline{A = 1} \qquad 2B = 6(1)$

$\underline{B = 3}$

$y_p = x^2 + 3x \qquad y = c_1 + c_2e^x + c_3e^{2x} + x^2 + 3x$

$y' = c_2e^{x} + 2c_3e^{2x} + 2x + 3$

$y'' = c_2e^{x} + 4c_3e^{2x} + 2$

- $0 = c_1 + c_2e^{0} + c_3e^{2(0)} + (0)^2 + 3(0)$

$c_1 + c_2 + c_3 = 0$

- $3 = c_2e^{0} + 2c_3e^{0} + 2(0) + 3$

$c_2 + 2c_3 = 0$ $\qquad c_2 = -2c_3$

- $4 = c_2e^{0} + 4c_3e^{0} + 2$

$2 = c_2 + 4c_3$

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$2 = -2c_3 + 4c_3$

$2 = 2c_3$

$\underline{c_3 = 1}$ $\qquad c_2 = -2(1)$ $\qquad c_1 + -2 + 1 = 0$

$\underline{c_2 = -2}$ $\qquad \underline{c_1 = 1}$

$$\underline{y = 1 - 2e^{x} + e^{2x} + x^2 + 3x}$$

ii- $x^2y'' - xy' = x^3 e^x \qquad y(1)=0, \quad y'(1)=3e$

cauchy-euler equation

$y = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1)x^{m-2}$

$\cancel{x^2} m(m-1) x^m \cdot \cancel{x^{-2}} - \cancel{x} m x^m \cdot \cancel{x^{-1}} = 0$

$x^m (m(m-1) - m) = 0$

$m^2 - m - m = 0$

$m^2 - 2m = 0$

$m(m-2) = 0$

$m = 0 \;\&\; m = 2 \qquad y_c = c_1 y_1 + c_2 y_2$

FS $\quad x^0 \;\&\; x^2 \qquad y_c = c_1 + c_2 x^2$

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$y = y_c + y_p$

$y_p = d_1 y_1 + d_2 y_2 = d_1 + d_2 x^2$

insight club

$W = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} = 2x - 0 = 2x$

$W_1 = \begin{vmatrix} 0 & x^2 \\ x^3e^x & 2x \end{vmatrix} = -x^5 e^x = -x^5 e^x$

$W_2 = \begin{vmatrix} 1 & 0 \\ 0 & x^3e^x \end{vmatrix} = x^3e^x - 0 = x^3 e^x$

$d_1' = -\dfrac{x^5 e^x}{2x} = -\dfrac{x^4 e^x}{2} \qquad d_2' = \dfrac{x^3 e^x}{2x} = \dfrac{x^2 e^x}{2}$

$\int d_1' = d_1$

$-\frac{1}{2}\int x^4 e^x =$

$\int f g' = fg - \int f' g$

$\int x^4 e^x = x^4 e^x - \int 4x^3 e^x$

$\int 4x^3 e^x = 4x^3 e^x - \int 12x^2 e^x$

$\int 12x^2 e^x = 12x^2 e^x - \int 24x e^x$

$\int 24 x e^x = 24xe^x - \int 24 e^x = 24xe^x - 24e^x$

$-\frac{1}{2}\int x^4 e^x = -\frac{1}{2}(x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x) \cdot -\frac{1}{2}$

$= -\frac{x^4e^x}{2} + 2x^3e^x - 6x^2e^x - 12xe^x - 12e^x$

$\int d_2' = \int \frac{x^2e^x}{2} = \frac{1}{2}\int x^2e^x$

$\int x^2e^x = x^2e^x - \int 2xe^x$

$\int 2xe^x = 2xe^x - \int 2e^x$

$= 2xe^x - 2e^x$

$\int x^2e^x = x^2e^x - 2xe^x + 2e^x$

$yp = \cancel{2x^3} \cdot -\frac{x^4e^x}{2} + 2x^3e^x - 6x^2e^x - 12xe^x - 12e^x + x^4e^x - 2x^3e^x + 2x^2e^x$

$yp = \frac{x^4e^x}{2} - 4x^2e^x - 12e^x - 12xe^x$

$-\frac{88}{2} + \frac{1}{2}$

$y = c_1 + c_2x^2 - 4x^2e^x - 12e^x + \frac{x^4e^x}{2} - 4x^2e^x - 12e^x - 12xe^x$

$0 = c_1 + c_2 - 4e - 12e + \frac{1}{2}e - 4e - 12e - 12e$

$0 = c_1 + c_2 - 8e - 36e + \frac{1}{2}e$

$0 = c_1 + c_2 - 44e + \frac{1}{2}e$

$c_1 + c_2 = \frac{87}{2}e$

$y' = 2c_2x - 8xe^x - 4x^2e^x - 12e^x + 2x^3e^x + \frac{x^4}{2}e^x - 8xe^x - 4x^2e^x - 12e^x - 12e^x - 12xe^x$

$3e = 2c_2 - 8e - 4e - 12e + 2e \ldots\ldots$

-3

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Problem 3

(10 points) Assume $g(y)$ is a differentiable function with a continuous derivative. Consider the following two equations:

$$\begin{aligned} y'(x) &= y(x) + g(y(x)) \\ y(x) &= \int_0^x e^{x-u} g(y(u))du \end{aligned}$$

Suppose $y_1(x)$ and $y_2(x)$ are two functions such that y_1 satisfies the first equation, y_2 satisfies the second one and moreover $y_1(0) = y_2(0) = 0$. Which of the following is true: $y_1(x) \geq y_2(x)$ or $y_2(x) \geq y_1(x)$?

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