Exercise 6 (15 pts) Solve the DE: $(3x+1/2)y^2\frac{dy}{dx}+\frac{3}{2x^2}+y^3=0$

Solution: Divide by y^2:

$$\left(3x+\frac{1}{2}\right)\frac{dy}{dx}+\frac{3}{2x^2y^2}+\frac{y^3}{y^2}=0$$

$$\left(3x+\frac{1}{2}\right)y'+y=\frac{3}{2x^2}y^{-2} \Rightarrow \boxed{y'+\frac{1}{(3x+1/2)}y=\frac{3}{2x^2(3x+\frac{1}{2})}y^{-2} \quad \text{Bernoulli with } n=-2}$$

Both sides Multiply y^2 $\Longrightarrow$ $y^2y'+\frac{1}{(3x+\frac{1}{2})}y^3=\frac{3}{2x^2(3x+1/2)}$

$u=y^{1-n}=y^{1-(-2)}=y^3$

$\Rightarrow u'=3y'y^2$

$\Rightarrow \frac{1}{3}u'=y'y^2$

Replace $\longrightarrow$ $\frac{1}{3}u'+\frac{1}{(3x+1/2)}u=\frac{3}{2x^2(3x+1/2)}$

$\longrightarrow$ $u'+\frac{3}{(3x+1/2)}u=\frac{9}{2x^2(3x+1/2)}$

$$I(x)=e^{\int\frac{3}{3x+1/2}dx}=e^{\ln(3x+1/2)}=(3x+1/2)$$

$$\left(3x+\frac{1}{2}\right)u'+\cancel{\left(3x+\frac{1}{2}\right)}\cdot\frac{3}{\cancel{(3x+1/2)}}u=\cancel{\left(3x+\frac{1}{2}\right)}\cdot\frac{9}{2x^2\cancel{(3x+1/2)}}$$

$$\left(3x+\frac{1}{2}\right)u'+3u=\frac{9}{2}x^{-2} \Rightarrow \left[\left(3x+\frac{1}{2}\right)u\right]'=\frac{9}{2}x^{-2}$$

$$\Rightarrow \left(3x+\frac{1}{2}\right)u=\int\frac{9}{2}x^{-2}dx=\frac{-9}{2}x^{-1}+C$$

$$\Rightarrow \boxed{\left(3x+\frac{1}{2}\right)y^3=-\frac{9}{2}x^{-1}+C}$$

Exercise 7 (15 pts) Multiply both sides of the differential equation by an appropriate factor to make it exact. Then Solve it: $(3xy^2 + 2x^2y)dy + (2y^3 + 3xy^2)dx = 0$

Solution: First divide by y: $(3xy + 2x^2)\,dy + (2y^2 + 3xy)\,dx = 0$

$u = \frac{y}{x}$ Homogeneous of degree 2

$\boxed{y = ux} \Rightarrow \boxed{dy = u\,dx + x\,du}$ $\quad \left(3x(ux) + 2x^2\right)(u\,dx + x\,du) + \left(2(ux)^2 + 3(x)(ux)\right)dx = 0$

$$\Rightarrow \left[3x^3u + 2x^3\right]du + \left[3x^2u^2 + 2x^2u + 2u^2x^2 + 3x^2u\right]dx = 0$$

$$\xrightarrow{\text{Divide } x^2} \left[3xu + 2x\right]du + \left[3u^2 + 2u + 2u^2 + 3u\right]dx = 0$$

$$\Rightarrow x\left[3u + 2\right]du + 5\left[u^2 + u\right]dx = 0$$

$$\Rightarrow \frac{3u+2}{5(u^2+u)}\,du + \frac{dx}{x} = 0 \Rightarrow \frac{3u+2}{5(u^2+u)}\,du = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{5}\int \frac{3u+2}{u^2+u}\,du = \int -\frac{dx}{x} \Rightarrow \frac{1}{5}\int\left(\frac{2}{u} + \frac{1}{u+1}\right)du = -\ln x + C$$

$$\Rightarrow \frac{2}{5}\ln(u) + \frac{1}{5}\ln(u+1) = -\ln x + C$$

$$\Rightarrow \frac{2}{5}\ln\left(\frac{y}{x}\right) + \frac{1}{5}\ln\left(\frac{y}{x} + 1\right) = -\ln x + C$$

$$\frac{3u+2}{u^2+u} = \frac{3u+2}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1} = \frac{Au + A + Bu}{u(u+1)} = \frac{(A+B)u + A}{u(u+1)}$$

$$\Rightarrow \begin{cases} A + B = 3 \\ A = 2 \end{cases} \Rightarrow \boxed{B = 1},\ A = 2 \Rightarrow \frac{3u+2}{u^2+u} = \frac{2}{u} + \frac{1}{u+1}$$