

(04NOV2016) Lecture 24

(...cntd) Boundary:

$$g(x) = x^2 - 3x + 10, g'(x) = 0 \Rightarrow 2x - 3 = 0, x = \frac{3}{2}; -3, \frac{3}{2}, 3$$

$$h(x) = 19 - 3x - 2\sqrt{9-x^2}, h'(x) = -3 + \frac{2x}{\sqrt{9-x^2}}, h'(x) = 0 \therefore \frac{2x}{\sqrt{9-x^2}} = 3 \quad (x > 0)$$

$$\Rightarrow \frac{4x^2}{9-x^2} = 9 \Rightarrow x = \pm \frac{9}{\sqrt{13}}, -3 \leq x \leq 3 \therefore -3, \frac{9}{\sqrt{13}}, 3$$

Check value of all the points:

$$f\left(\frac{3}{2}, 1\right) = \frac{9}{4} + 1 - \frac{9}{2} - 2 + 10 = 6.75$$

$$g(-3) = f(-3, 0) = 9 + 9 + 10 = 28$$

$$g\left(\frac{3}{2}\right) = f\left(\frac{3}{2}, 0\right) = \frac{9}{4} - \frac{9}{2} + 10 = 7.75$$

$$g(3) = 9 - 9 + 10 = 10$$

$$h\left(\frac{9}{\sqrt{13}}\right) \approx 8.18$$

Global maximum: 28
Global minimum: 6.75
This means:
 $6.75 \leq x^2 + y^2 - 3x - 2y + 10 \leq 28$

Alternatively:

The arc \widehat{ACB} is parametrised by $x = 3\cos t, y = 3\sin t \quad 0 \leq t \leq \pi$

$$\Rightarrow f(3\cos t, 3\sin t) = 9 - 9\cos t - 6\sin t + 10$$

$$\therefore g(t) = 19 - 9\cos t - 6\sin t \quad 0 \leq t \leq \pi$$

$$g'(t) = 9\sin t - 6\cos t \Rightarrow g'(t) = 0 \therefore \tan t = \frac{2}{3} \Rightarrow t_0 = \tan^{-1}\left(\frac{2}{3}\right)$$

Note: Given t and $\alpha = x$

$$\cos^2 \alpha = \frac{1}{1+x^2}, \sin^2 \alpha = \frac{x^2}{1+x^2}$$

compact set:

Domain which is closed and bounded

Extreme values with constraints

Given $f(x_1, x_2, \dots, x_n), g_1, g_2, g_3, \dots, g_n$

Find extreme values of f subject to $g_i(x_1, x_2, \dots, x_n) = 0$

Example: $f(x, y) = xy$

Constraint: $\frac{x^2}{4} + y^2 = 1$

$f(x, y, z) = x^2 + y^2 + z^2$

constraints $\begin{cases} x+y+z=1 \\ x^2+y^2=4 \end{cases}$

$$g_2(x_1, x_2, \dots, x_n) = 0 \quad \text{etc}$$

$$g_M(x_1, x_2, \dots, x_n)$$

(07NOV2016) Lecture 25

Method of Lagrange multipliers to find extreme values of functions on level curves or surfaces.

- Find the extreme values of the function $f(x, y) = xy$ over the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

- Level curve is $g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$

Theorem: If there is a point on the level curve where f attains an extreme value, then the gradient of f at that point:

$$\vec{\nabla}f(x, y) = \lambda \vec{\nabla}g(x, y) \text{ where } \lambda \text{ is a constant}$$

Method: compute $\vec{\nabla}f$, $\vec{\nabla}g$ and solve the 3 equations for x, y & λ . Obtain the point (x, y) and decide if you have a maximum or a minimum.

Solution: $\vec{\nabla}f(x, y) = y\vec{i} + x\vec{j}$ } Since $\vec{\nabla}f(x, y) = \lambda \vec{\nabla}g(x, y)$

$$\vec{\nabla}g(x, y) = \frac{x}{4}\vec{i} + y\vec{i} \quad \left. \right\} : y = \frac{\lambda x}{4}, x = \lambda y \text{ and } \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

Note: If $\lambda = 0$, x & y would be zero which is impossible according to the third equation. $\therefore \lambda \neq 0$.

$$\Rightarrow xy = \frac{\lambda x^2}{4}, xy = \lambda y^2 \quad \therefore \frac{\lambda x^2}{4} = \lambda y^2 \Rightarrow \frac{x^2}{4} = y^2 \quad (\text{since } \lambda \neq 0)$$

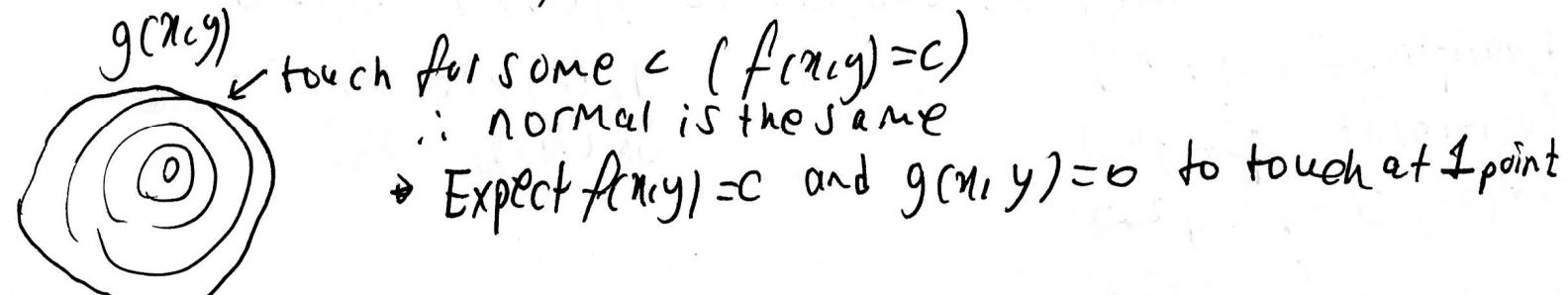
$$\text{thus } \frac{y^2}{2} + \frac{y^2}{2} - 1 = 0, y = \pm 1 \text{ and } x = \pm 2 \quad (4 \text{ points})$$

A(2, 1), B(-2, 1), C(2, -1), D(-2, -1)

check for all the points:

$$\begin{cases} f(A) = 2 \\ f(D) = 2 \end{cases} \quad \begin{cases} f(c) = -2 \\ f(B) = -2 \end{cases} \quad \begin{aligned} & f(x, y)_{\max} = 2 \\ & f(x, y)_{\min} = -2 \end{aligned}$$

Proof of Theorem (idea)



[06 JAN 2017]

(09 NOV 2016) Lecture 26

- Will study:

$$\iint_D f(x, y) dA$$

$$\iiint_R f(x, y, z) dV$$

How is $\int_a^b f(x) dx$ defined?

Answer:

$$f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \dots + f(c_n)\Delta x_n$$

- Finding primitives:

$$\int x^n dx = \frac{x^{n+1}}{n+1} \quad n \neq -1$$

$$\int x^{-1} dx = \ln x$$

$$\int e^{ax} dx = \frac{e^{ax}}{a}$$

$$\int \cos(\omega x) dx = \frac{\sin \omega x}{\omega}$$

$$\int \sin(\omega x) dx = -\frac{\cos \omega x}{\omega}$$

- Riemann sum:

$$\sum f(x_i, y_i) \Delta x_i \Delta y_i$$

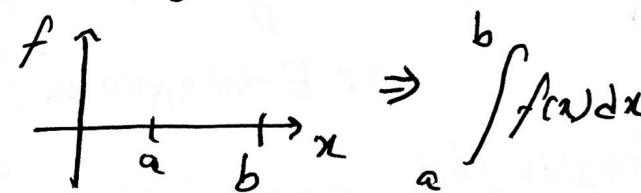
If limit of such sums exist it is called the double integral off over D & Denoted as: $\iint_D f(x, y) dA$

Multiple Integrals

- Applications:

Finding: areas, volumes, moments, centers of mass/gravity, moment of inertia, higher moments, (used in statistics)

First we go back to one variable:



$$\therefore \int_a^b f(x) dx = \text{limit of Riemann sums}$$

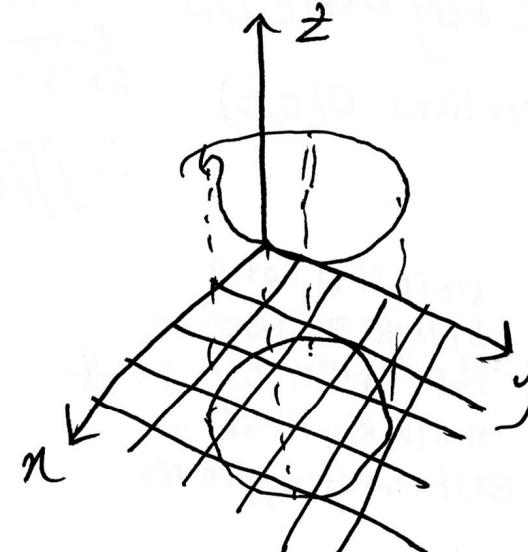
• How do we calculate it?
Fundamental theorem of calculus
(Newton)

\Rightarrow First find a primitive F off, that is $F'(x) = f(x)$

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

$$\text{ex: } \int_1^2 x^3 dx = \frac{x^4}{4} \Big|_1^2 = \frac{24}{4} - \frac{1}{4} = 3.75$$

• Let f be a function of 2 variables (x, y) defined on the domain D on x-y plane.



[06JAN2016]

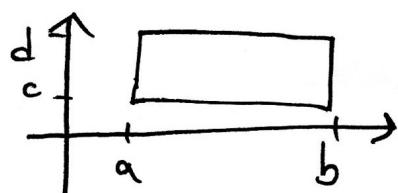
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How to compute double integrals?

Suppose D is a rectangle,

$$D = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$$

and f is defined on D



$$\iint_D f(x,y) dA = \int_a^b \left(\int_c^d f(x,y) dy \right) dx$$

Example:

$$D = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq 2\} \Rightarrow \int_0^1 \left[\int_0^2 y \sin x dy \right] dx = -\cos x \Big|_0^1 = -\cos 1 + \cos 0$$

$$f(x,y) = x^3 y^5$$

$$\text{then } \iint_D f(x,y) dA = \int_0^2 \left(\int_0^1 x^3 y^5 dx \right) dy$$

$$= \int_0^2 \frac{y^5 x^4}{4} \Big|_0^1 dy = \int_0^2 \left(\frac{y^4}{4} - 0 \right) dy$$

$$= \frac{y^5}{24} \Big|_0^2 = \frac{2^5}{24}, \text{ or it can also be done as:}$$

$$\iint_D x^3 y^5 dy dx \text{ by}$$

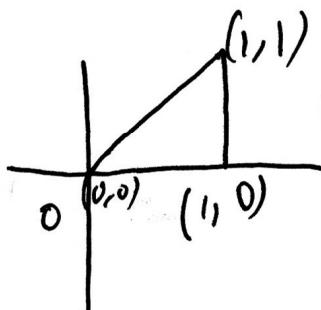
Fubini theorem:

Evaluate $\iint_D \sin x dy dx$ where D is

the triangle of vertices $O(0,0)$

$A(1,0)$, $B(1,1)$

- Sketch of D



since it is not a triangle \Rightarrow describe the region o.e south-north approach or east-west approach

Enter: $OA (y=0, 0 \leq x \leq 1)$

Exit: $OB (y=x, 0 \leq x \leq 1)$

or

Enter: $OB (x=y, 0 \leq y \leq 1)$

Exit: $AB (x=1, 0 \leq y \leq 1)$

S-N approach

$$\therefore \iint_D \frac{\sin x}{x} dA = \int_0^1 \left(\int_0^x \frac{\sin x}{x} dy \right) dx$$

$$\text{or E-W approach: } \int_0^1 \left(\int_y^1 \frac{\sin x}{x} dx \right) dy$$

choose S-N approach because it is easier
(in fact E-W approach cannot be done)

$$\int_0^1 \frac{y \sin x}{x} dx \Big|_0^1 = -\cos x \Big|_0^1 = -\cos 1 + \cos 0$$

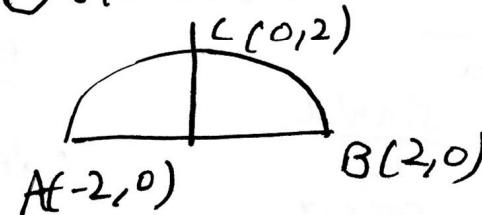
$$= 1 - \cos 1$$

• Consider $\iint_D f(x,y) dA$ where

$$D = \{(x,y) \mid x^2 + y^2 \leq 4, y \geq 0\}$$

Set up its limit of integration

① Sketch D :



$A(-2,0)$

$B(2,0)$

• S-N approach

$$\text{Enter: } AB (y=0, -2 \leq x \leq 2)$$

$$\text{Exit: } ACB (y=\sqrt{4-x^2}, -2 \leq x \leq 2)$$

$$\therefore \iint_D f(x,y) dA = \int_{-2}^2 \int_{\sqrt{4-x^2}}^{f(x,y)} f(x,y) dy dx$$

E-W approach

Enter \widehat{AC} ($x = -\sqrt{4-y^2}, 0 \leq y \leq 2$)
 Exit: \widehat{CB} ($x = \sqrt{4-y^2}, 0 \leq y \leq 2$)

$$\iint_D f(x,y) dA = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y) dx dy$$

(11 NOV 2016) lecture 27

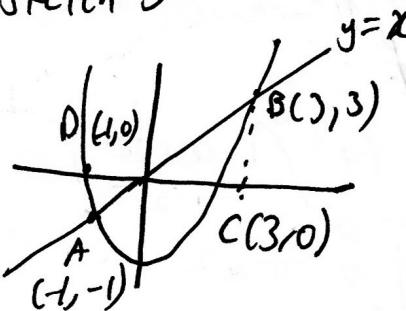
Note:

S-N approach $\int_a^b \int_x^y f(x,y) dy dx$

E-W approach $\int_a^b \int_y^x f(x,y) dx dy$

Example: Evaluate the integral if where $f(x,y) = x^2 + y$ over the region bounded by $y = x$ and $y = x^2 - x - 3$

① Sketch D



Find intersection of $y = x$ and $y = x^2 - x - 3$

$$\therefore (-1, -1), (3, 3)$$

For S-N approach enter from

$$y = x^2 - x - 3, -1 \leq x \leq 3$$

and exit: $y = x, -1 \leq x \leq 3$

$$\begin{aligned} & \int_{-1}^3 \left(\int_{x^2-x-3}^x (x^2+y) dy \right) dx \quad (\text{getting up limits of integration}) \end{aligned}$$

$$\Rightarrow \int_{-1}^3 \left[x^2y + \frac{y^2}{2} \right]_{x^2-x-3}^x dx = \text{etc.}$$

E-W approach

find minima: $\frac{dy}{dx} = 0 \therefore \left(\frac{1}{2}, -\frac{13}{4} \right)$
 Enter from 2 different places (see diagram)
 for $-\frac{13}{4} \leq y \leq -1, y = x^2 - x - 3$
 (must be written in terms of x)

$$\Rightarrow x^2 - x - 3 - y = 0$$

$$x = \frac{1 \pm \sqrt{13+4y}}{2}$$

$$\therefore \begin{aligned} \text{Enter } x &= \frac{1 - \sqrt{13+4y}}{2}, & -\frac{13}{4} \leq y \leq -1 \\ \text{Exit } x &= \frac{1 + \sqrt{13+4y}}{2}, & -\frac{13}{4} \leq y \leq -1 \end{aligned}$$

And: Enter $x = y, -1 \leq y \leq 3$

$$\text{Exit } x = \frac{1 + \sqrt{13+4y}}{2}, -1 \leq y \leq 3$$

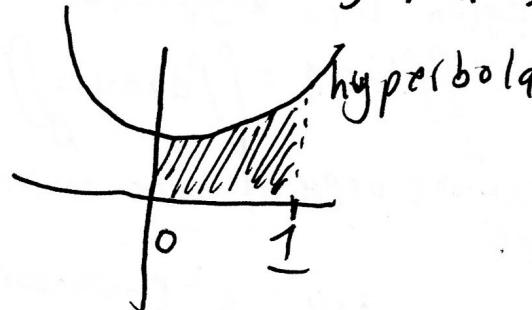
$$\begin{aligned} \iint_D f(x,y) dA &= \int_{-1}^3 \left(\int_{\frac{1-13+4y}{2}}^{\frac{1+13+4y}{2}} (x^2 + y) dx \right) dy \\ &+ \int_{-1}^3 \left(\int_{\frac{1+13+4y}{2}}^{\frac{1-13+4y}{2}} (x^2 + y) dx \right) dy \end{aligned}$$

Problem: Sketch the region of integration

$$\int_0^1 \int_0^{\sqrt{x^2+1}} \sin(xy) dy dx$$

Answer: Enter $y = 0, 0 \leq x \leq 1$

$$\text{Exit } y = \sqrt{x^2+1}, 0 \leq x \leq 1$$



Problem: Interchange the order of integration:

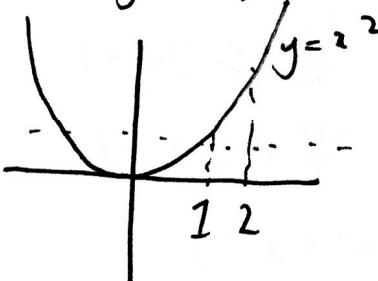
$$\int_0^2 \left(\int_0^{x^2} x^2 y dy \right) dx$$

Must sketch D first!!!

S-N approach:

Enter: $y=0, 0 \leq x \leq 2$

Exit: $y=x^2, 0 \leq x \leq 2$



E-W approach

Enter $x=1, 0 \leq y \leq 1$

Exit $x=2, 0 \leq y \leq 1$

Enter $x=\sqrt{y}, 1 \leq y \leq 4$

Exit $x=2, 1 \leq y \leq 4$

$$\begin{aligned} \therefore \int_1^2 \int_0^{x^2} x^2 y dy dx &= \int_0^1 \int_{\sqrt{y}}^2 x^2 y dy dx \\ &+ \int_1^{\sqrt{y}} \int_{\sqrt{y}}^2 x^2 y dy dx \end{aligned}$$

(This is called interchange of limiting operations ie $\int \lim = ? \lim \int$)

$$\int \Sigma = ? \int \Sigma \text{ etc..}$$

Applications: Areas

Given a region D in x-y plane

$$\text{Area of } D = \iint_D 1 \cdot dA = \int_0^R \int_0^{f(x)} dy dx = \iint_D dy dx$$

Example: Find the area of disc centre (0,0) radius $a > 0$.

Sketch

$$x^2 + y^2 = a^2 \text{ (boundary)}$$



S-N approach

Enter: $y = -\sqrt{a^2 - x^2}, -a \leq x \leq a$

Exit: $y = \sqrt{a^2 - x^2}, -a \leq x \leq a$

$$\therefore \iint_D dy dx = \int_{-a}^a \left(\int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} dy \right) dx$$

$$= 2 \int_a^a \sqrt{a^2 - x^2} dx = 4 \int_0^a \sqrt{a^2 - x^2} dx$$

To evaluate this integral, since it is an even function, try to change what is under the square root to be a perfect square:

$$\therefore \cos^2 \theta + \sin^2 \theta = 1$$

$$\Rightarrow a^2 \sin^2 \theta = a^2 - a^2 \cos^2 \theta$$

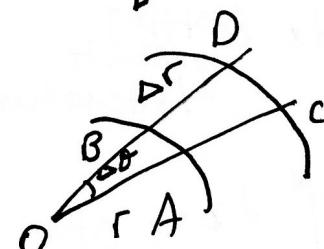
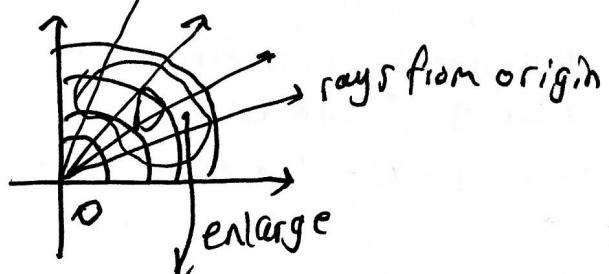
Put $x = a \cos \theta \therefore dx = -a \sin \theta d\theta$

$$\text{So, } 4 \int_0^a \sqrt{a^2 - x^2} dx = 4 \int_0^{\pi/2} \sqrt{a^2 - a^2 \cos^2 \theta} (-a \sin \theta) d\theta$$

$$= 4 \int_0^{\pi/2} a^2 \sin^2 \theta d\theta = 4a^2 \int_0^{\pi/2} \frac{1 - \cos 2\theta}{2} d\theta$$

$$= 2a^2 \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = \pi a^2$$

(14 N D 2016) [07 JAN 2017] deetare 28
double integrals in polar coordinates



$$\text{Area} = r^2 \theta - r^2 \sin \theta$$

$$\text{Area of sector} = \frac{\alpha a^2}{2}$$

$$\therefore \text{area} = \frac{1}{2}(r+\Delta r)^2 \Delta\theta - \frac{1}{2}r^2 \Delta\theta \\ = r \Delta r \Delta\theta + \frac{1}{2}(\Delta r)^2 \Delta\theta$$

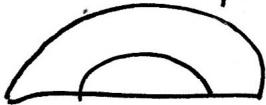
A Riemann sum:

$$\sum f(r_k, \theta_k) \cdot r_k \Delta r_k \Delta\theta + \text{terms with } (\Delta r_k)^2 \Delta\theta_k$$

$$\text{in the limit: } \iint f(r, \theta) r dr d\theta$$

• How to set up limits of integration

i)

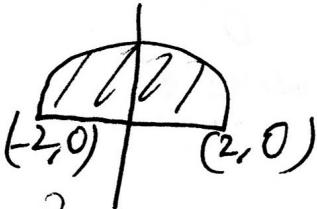


$$\text{Enter: } r=1, 0 \leq \theta \leq \pi$$

$$\text{Exit: } r=\sqrt{2}, 0 \leq \theta \leq \pi$$

$$\therefore \iint_0^{\sqrt{2}} f(r \cos\theta, r \sin\theta) r dr d\theta$$

Examples:

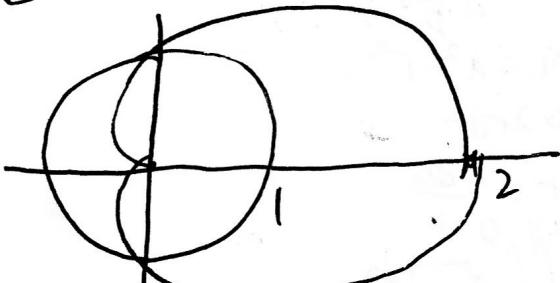


$$\iint_0^{\pi/2} f(r \cos\theta, r \sin\theta) r dr d\theta$$

More examples:

Find area of region inside cardioid $r=1+\cos\theta$ and outside circle $r=1$

① Sketch:



$$\text{Area} = \iint_D dA$$

Describe D

$$\text{Enter: } r=1, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$\text{Exit: } r=1+\cos\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$\iint_{-\pi/2}^{\pi/2} r dr d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2} \left| \frac{1+\cos\theta}{2} \right|^2 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{(1+\cos\theta)^2}{2} - \frac{1}{2} \right) d\theta$$

$$= \frac{1}{2} \int (2\cos\theta + \cos^2\theta) d\theta$$

$$= \frac{1}{2} \int (2\cos\theta + \frac{1+\cos2\theta}{2}) d\theta$$

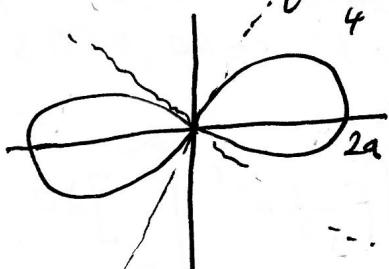
$$= \frac{1}{2} \left[2\sin\theta + \frac{\theta}{2} + \frac{\sin2\theta}{4} \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2} \left[(2 + \frac{\pi}{4} + 0) - (-2 - \frac{\pi}{4} - 0) \right]$$

$$= 2 + \frac{\pi}{4}$$

• Find area bounded by the lemniscate $r^2 = 4a^2 \cos 2\theta$

① sketch region



By symmetry:

Area = 4 times area of 1st quadrant.

Describe D:

$$\text{Enter: } r=0, 0 \leq \theta \leq \frac{\pi}{4}$$

$$\text{Exit: } r=2a \sqrt{\cos 2\theta}, 0 \leq \theta \leq \frac{\pi}{4}$$

$$\text{Area} = 4 \int_0^{\pi/4} \int_0^{2a\sqrt{\cos 2\theta}} r dr d\theta = 4 \int_0^{\pi/4} \frac{1}{2} \left| \frac{2a\sqrt{\cos 2\theta}}{2} \right|^2 d\theta$$

$$= 2 \int_0^{\frac{\pi}{4}} 4a^2 \cos 2\theta \, d\theta$$

$$= 8 \int_0^{\frac{\pi}{2}} a^2 \cos 2\theta \, d\theta$$

$$= 8 \left[a^2 \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} = 4a^2$$

Historical: Evaluate

$$\int_0^\infty e^{-x^2} dx$$

Solution: Let $I = \int_0^\infty e^{-x^2} dx$
and evaluate I^2 .

$$\therefore I^2 = I \cdot \int_0^\infty e^{-x^2} dx$$

$$I^2 = \int_0^\infty e^{-x^2} \cdot I = \int_0^\infty e^{-x^2} \left(\int_0^\infty e^{-y^2} dy \right) dx$$

$$= \int_0^\infty \int_0^\infty e^{-x^2-y^2} dy dx$$

Region of integration

1st quadrant \Rightarrow in polar
 \Rightarrow Enter: $r=0, 0 \leq \theta \leq \frac{\pi}{2}$
 Exit: $r=\infty, 0 \leq \theta \leq \frac{\pi}{2}$

$$I^2 = \int_0^{\frac{\pi}{2}} \int_0^\infty r e^{-r^2} r dr d\theta = \int_0^{\frac{\pi}{2}} \left[-\frac{1}{2} e^{-r^2} \right]_0^\infty d\theta$$

$$= \frac{\pi}{2} \left[0 - \left(-\frac{1}{2} \right) \right] d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{4} = I^2$$

$$\Rightarrow I = \frac{\sqrt{\pi}}{2}$$

$$\therefore \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \Rightarrow \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

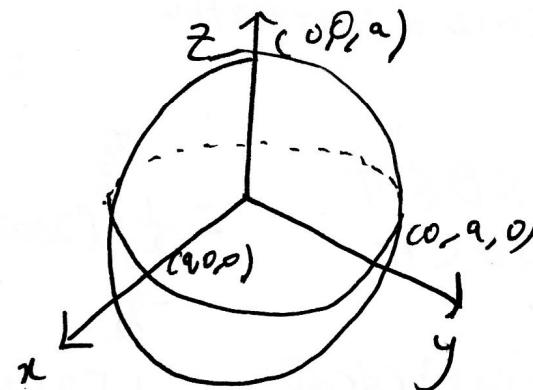
$$\therefore \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} dx = 1 \quad \text{This is the normal}$$

What about $\int_0^\infty e^{-x^3} dx$ or more generally $\int_0^\infty e^{-x^d} dx$? [no solutions]

(16NOV2016) lecture 29

- Find the volume of a sphere of radius a .
- We may assume that centre of sphere is at the origin O so its equation:

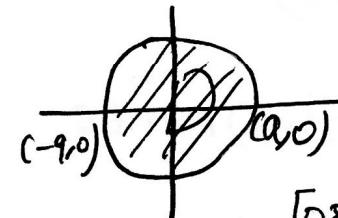
$$x^2 + y^2 + z^2 = a^2$$



Volume total = 2 × Volume of the upper hemisphere = $2 \iint \sqrt{a^2 - x^2 - y^2} dA$

Need to sketch D in the x-y plane.

D is a disc centre O radius a.



[08JAN2016] 12:17A.M

In polar:

Enter $r=0, 0 \leq \theta \leq 2\pi$
 Exit $r=a, 0 \leq \theta \leq 2\pi$

$$\text{Volume} = 2 \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2} r dr d\theta$$

$$\text{Let } u = a^2 - r^2$$

$$\Rightarrow du = -2r dr$$

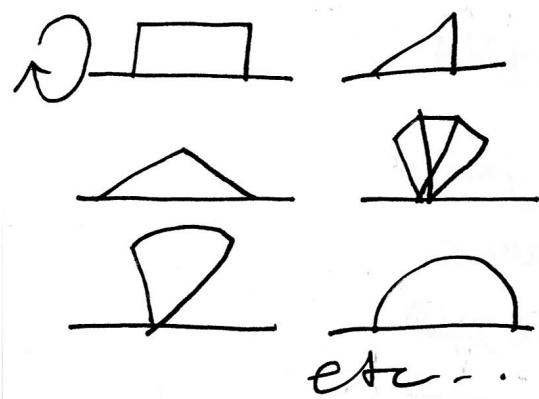
$$r dr = -\frac{du}{2}$$

$$\therefore 2 \int_0^{2\pi} \int_0^{a^2} \sqrt{u} \frac{du}{2}$$

Gaussian distribution

$$\begin{aligned}
 &= 2 \iint_D \frac{1}{2} u^{a^2} du = \int_0^{2\pi} \int_0^{\frac{2\pi}{3}} u^{\frac{3}{2}} \Big|_0^{a^2} d\theta \\
 &= \int_0^{2\pi} \frac{2}{3} a^3 d\theta = \frac{2}{3} a^3 \int_0^{2\pi} d\theta \\
 &= \frac{2}{3} a^3 (2\pi) = \frac{4}{3} \pi a^3
 \end{aligned}$$

• Archimedes' way:



Triple integrals:

3 variables

$$w = f(x, y, z)$$

R is a solid region in 3-space

$$\Rightarrow \iiint_R f(x, y, z) dV$$

To evaluate a triple integral:

① Sketch D , it will be a solid in space.



Sketch a surface

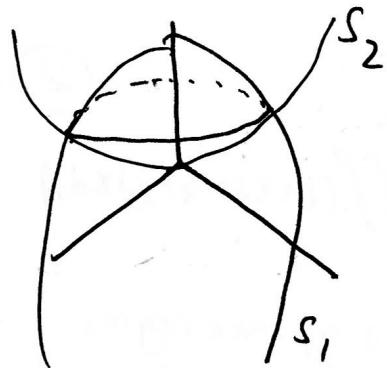
② Describe

Enter: surfaces,
 $z = g_1(x, y)$

Exit: surface S_2 ($z = g_2(x, y)$)
(x, y) is in region D , projection of
the solid R on the x - y plane

$$\begin{aligned}
 &\therefore \iiint_R f(x, y, z) dV = \iint_D f(x, y, g_2(x, y)) dz dA \\
 &\text{R} \qquad \qquad \qquad D, g_2(x, y) \\
 &\cdot \text{Volume of } R = \iiint_R dV
 \end{aligned}$$

Example: $\iiint_R dV$ where R is a solid bounded
above R by S_1 : $z = 8 - x^2 - y^2$ and below
by S_2 : $z = x^2 + y^2$ (paraboloid)



R :

$$\text{Enter: } z = x^2 + y^2 \quad (x, y) \text{ in } D$$

$$\text{Exit: } z = 8 - x^2 - y^2 \quad (x, y) \text{ in } D$$

$$D: z = z$$

$$x^2 + y^2 = 8 - x^2 - y^2$$

$$\therefore x^2 + y^2 = 4$$

$\Rightarrow D$ is the disc in x - y plane
bounded by the circle $x^2 + y^2 = 4$

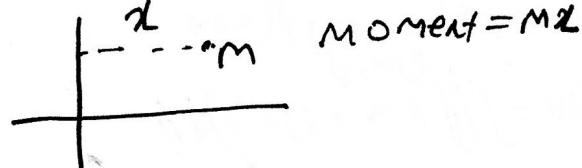
$$\begin{aligned}
 \iiint_R dV &= \iint_D \left(\int_{x^2+y^2}^{8-x^2-y^2} dz \right) dA \\
 R \qquad D \qquad x^2+y^2
 \end{aligned}$$

$$\begin{aligned}
 &= \int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz dy dx = \iint_D (8 - 2x^2 - 2y^2) dA
 \end{aligned}$$

Describe in polar:

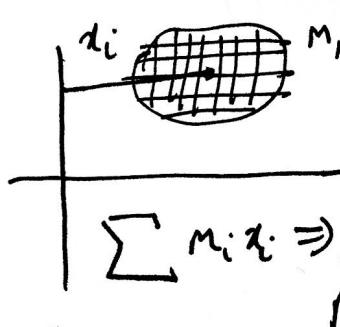
$$\begin{aligned}
 &= \int_0^{2\pi} \int_0^2 (8 - 2r^2) r dr d\theta \quad \text{etc.} \\
 0 \qquad 0
 \end{aligned}$$

(18NOV2016) lecture 30
Moments, centre of mass, mass



In general:

$$\sum m_i x_i \Rightarrow \iint_D x f(x, y) dA$$



To sum up, if D is a plane region with density function $f(x, y)$ at (x, y)

then: Moment D w.r.t y-axis $\iint_D x f(x, y) dA$
and mass $= \iint_D f(x, y) dA$

$$\bar{y} = \frac{\iint_D x f(x, y) dA}{\iint_D f(x, y) dA}$$

Similarly moment w.r.t x-axis $\iint_D y f(x, y) dA$

Let R be a solid in 3-space D with a density function $f(x, y, z)$ at the point (x, y, z)

$$\text{Mass } M = \iiint_R f(x, y, z) dV$$

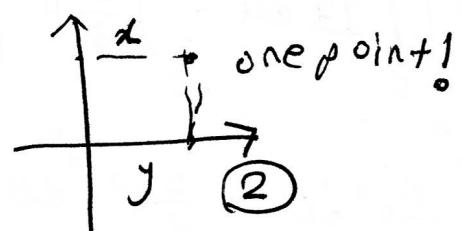
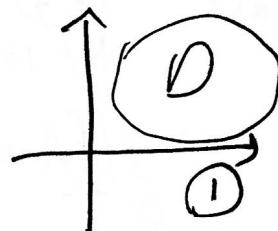
Moments in 3d

$$\iiint_R x f(x, y, z) dV = M_{yz}$$

$$\iiint_R y f(x, y, z) dV = M_{xz}$$

$$\iiint_R z f(x, y, z) dV = M_{xy}$$

called first moments (x', y', z')



• seek a point (\bar{x}, \bar{y}) so that if all the mass of D is concentrated at (\bar{x}, \bar{y}) , it will have the same moments as D .

$$① \cdot M_y = \iint_D x f(x, y) dA$$

$$M_y = \iint_D y f(x, y) dA$$

$$② \cdot M_{\bar{x}} = (m\bar{y})_{\text{y-axis}}$$

$$M\bar{y} = (m\bar{y})_{\text{z-axis}}$$

$$\bar{x} = \frac{\iint_D x f(x, y) dA}{\iint_D f(x, y) dA}$$

$$\bar{y} = \frac{\iint_D y f(x, y) dA}{\iint_D f(x, y) dA}$$

$$\bar{z} = \frac{\iint_D z f(x, y) dA}{\iint_D f(x, y) dA}$$

In 3-d the com of a solid R with density function $f(x, y, z)$ is a point $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \iiint_R x f(x, y, z) dV$$

$$\bar{y} = \frac{\iiint_R y f(x, y, z) dV}{\iiint_R f(x, y, z) dV} \text{ etc.}$$

Note: when asked for center of gravity (COG)
then $f(x, y) = 1$

Example: find COG of the solid bounded by the sphere $x^2 + y^2 + z^2 = a^2$ and the $x-y$ plane in the upper semi space.

$$\begin{aligned}
 &= \int_0^{2\pi} \int_0^a \left(\frac{\alpha^2 - r^2}{2} \right) r dr d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^a (\alpha^2 - r^2) r dr d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \frac{\alpha^2 r^2}{2} - \frac{r^4}{4} \Big|_0^a d\theta = \frac{1}{2} \int_0^{2\pi} \left(\frac{\alpha^4}{2} - \frac{a^4}{4} \right) d\theta \\
 &= \frac{a^4}{8} \int_0^{2\pi} d\theta = \frac{2\pi a^4}{8} = \frac{\pi a^4}{4}
 \end{aligned}$$

We need to compute 4 integrals over R . Similarly for $M_{yz} = 0$ by symmetry
 $M_{zx} = 0$

$$\text{Mass} = \iiint_R f(x, y, z) dV = \iiint_R dV \quad \because \text{coordinates of COG}$$

Describe R :

Enter at $z=0$, (x, y) inside disc

• f center $(0, 0)$, $r=a$

Exit $z=\sqrt{a^2-x^2-y^2}$, $1/1$

$$\text{Mass} = \int_D \int_0^a \left(\int_0^{\sqrt{a^2-x^2-y^2}} dz \right) dA$$

$$= \int_D \int_0^a \left(z \Big|_0^{\sqrt{a^2-x^2-y^2}} \right) dA = \int_D \sqrt{a^2-x^2-y^2} dA$$

Describe D in polar:

• Enter $r=0$, $0 \leq \theta \leq 2\pi$

Exit $r=a$, $0 \leq \theta \leq 2\pi$

$$M = \int_0^{2\pi} \int_0^a \sqrt{a^2-r^2} r dr d\theta$$

$$= \left(\frac{4}{3} \pi a^3 \right) \frac{1}{2} = \frac{2}{3} \pi a^3$$

The moments:

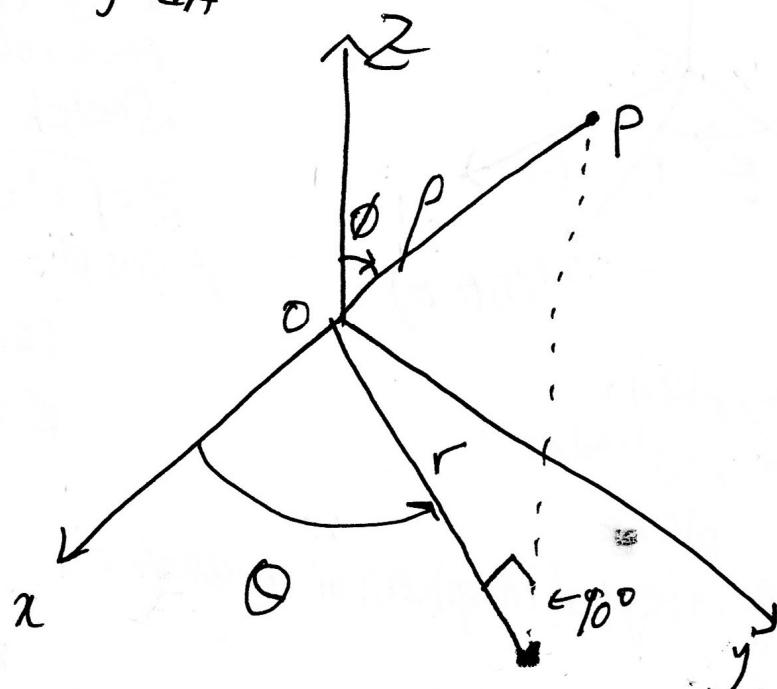
$$M_{xy} = \iiint_R z f(x, y, z) dV$$

$$= \iiint_R z dV = \iiint_D z dz dA$$

$$\begin{aligned}
 \bar{x} &= 0 \\
 \bar{y} &= 0 \\
 \bar{z} &= \frac{\pi a^4}{4} = \frac{3a}{8} \\
 \therefore & (0, 0, \frac{3a}{8})
 \end{aligned}$$

(21 NOV 2016) Lecture 31 [09 JAN 2017]
 Integration in spherical/cylindrical coordinates

spherical:



$$\sin\phi = \frac{r}{\rho}$$

$$\cos\phi = \frac{z}{\rho}$$

But $x = r\cos\theta = \rho\sin\phi\cos\theta$
 $y = r\sin\theta = \rho\sin\phi\sin\theta$
 $z = \rho\cos\phi$

Spherical coordinates:

$$(\rho, \theta, \phi)$$

$$\downarrow \quad \downarrow \quad \rightarrow 0 \leq \phi \leq \pi$$

$$0 < \rho < \infty \quad 0 \leq \theta \leq 2\pi$$

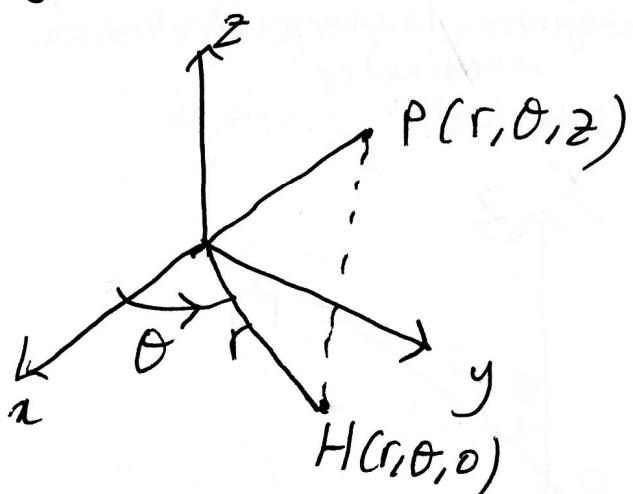
- $\rho = c$, sphere center zero radius c

- $\theta = d$, plane

- $\phi = \beta$, cone

- $x^2 + y^2 + z^2 = \rho^2$

- Cylindrical



- $z = c$, plane

- $r = d$, cylinder

- $\theta = d$, plane

- Triple integral in spherical coordinates:

$$\iiint_R f(x, y, z) dV$$

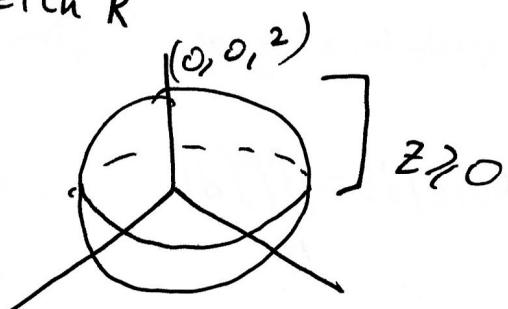
R

$$= \iiint_{R} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

Describe R in spherical coordinates

$$R = \{(x, y, z) : x^2 + y^2 + z^2 \leq 4, z \geq 0\}$$

- sketch R



Enter $\rho = 0, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq 2\pi$

Exit $\rho = 2, \dots, \dots$

$$\Rightarrow \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

- Ice cream cone

Let R be the region bounded by $x^2 + y^2 + z^2 = 4$ and $z = \sqrt{x^2 + y^2}$; $r^2 \geq 0$
 find volume of R

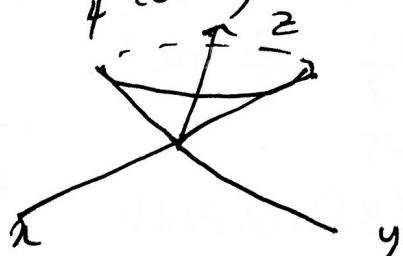
- Sketch R

$$z = \sqrt{x^2 + y^2}$$

$$\rho \cos\phi = \sqrt{r^2} = r = \rho \sin\phi$$

$$\tan\phi = 1$$

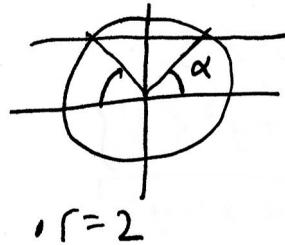
$$\phi = \frac{\pi}{4} \text{ (cone)}$$



Describe R:

In polar:

Enter: $\rho = 0, 0 \leq \theta \leq \frac{\pi}{4}, 0 \leq \theta \leq 2\pi$
 Exit: $\rho = 2, \theta = \frac{\pi}{4}$



Volume of R = $\iiint dV = \int \int \int \rho^2 \sin \theta d\rho d\theta d\theta$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \rho^3 \sin \theta \left| \begin{array}{l} \\ 3 \end{array} \right|^2 d\theta d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{8}{3} \sin \theta d\theta d\theta$$

$$\sqrt{3} = 2 \cos \alpha$$

$$\cos \alpha = \frac{\sqrt{3}}{2}$$

$$\alpha > \frac{\pi}{6} = \theta_1$$

$$\theta_2 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Enter: $r \sin \theta = 1, \frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$

$$y \uparrow \\ r = \frac{1}{\sin \theta}$$

Exit: $r = 2, \theta$

$$\cdot \int_{\frac{5\pi}{6}}^{\frac{2\pi}{3}} \int_0^2 r dr d\theta \text{ etc.}$$

$$\frac{\pi}{6} \cot \theta$$

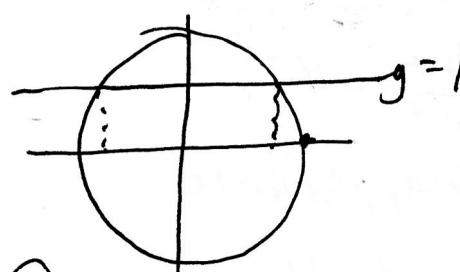
- Find Volume of the solid region R in space bounded below by plane $z=1$ and above by the sphere $x^2+y^2+z^2=4$
- Set up integral in cartesian cylindrical, and spherical.

(23 NOV 2016) Lecture 32

Exercises:

Find the area of the region in the x-y plane bounded below by $y=1$, above by $x^2+y^2=4$. Set up the integral in polar coordinates and also in Cartesian.

① Sketch D



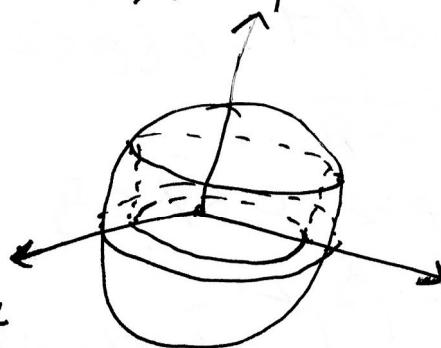
② In cartesian

Enter $y=1, -\sqrt{3} \leq x \leq \sqrt{3}$ ($\rho^2 \sin^2 \theta = 1$, $x^2+y^2=4$)

Exit: $y = \sqrt{4-x^2}$

$$+\sqrt{3} \int_{-\sqrt{3}}^{\sqrt{3}} \int_0^{\sqrt{4-x^2}} dA \text{ etc...}$$

$$-\sqrt{3} \int_1^{\sqrt{3}}$$



Cartesian:

Enter $z=1, (x,y) \in D$

Exit $z = \sqrt{4-x^2-y^2}$

$$\therefore \int \int \left(\int_1^{\sqrt{4-x^2-y^2}} dz \right) dA$$

$$P_{1,1}$$

To find D,

$$\text{put } z=1; x^2+y^2=3$$

$$D_1: \{(x, y) \mid x^2+y^2 \leq 3\}$$

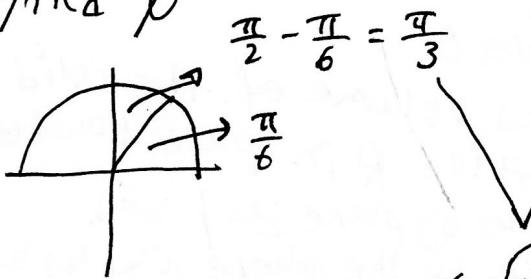
$$\Rightarrow \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{3-x^2}} dz dy dx$$

Cylindrical:

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \int_{4-r^2}^{\sqrt{3}} dz r dr d\theta$$

Spherical:

find ϕ



$$\text{Enter } z=1, 0 \leq \phi \leq \pi - \frac{\pi}{6} = \frac{\pi}{3}$$

$$\rho \cos \phi = 1$$

$$\rho = \frac{1}{\cos \phi}$$

$$\text{Exit: } \rho = 2, 0 \leq \phi \leq \frac{\pi}{3}$$

$$\therefore \int_0^2 \int_0^{\frac{\pi}{3}} \int_{\sec \phi}^2 \rho^2 \sin \phi d\rho d\phi d\theta$$

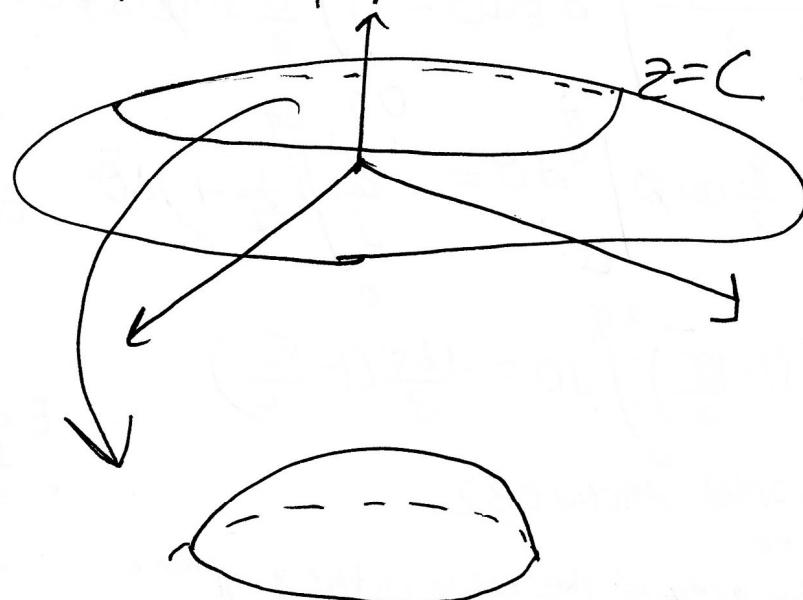
(28 NOV 2016) Lecture 32

find the volume of the region inside

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 4 \text{ and above plane } z=$$

$$\left(\frac{x}{2a} \right)^2 + \left(\frac{y}{2a} \right)^2 + \left(\frac{z}{2a} \right)^2 = 1$$

Sketch of R:



$$V = \iiint dV \Rightarrow \text{make change of variables}$$

$R \quad x=a\psi, y=b\psi, z=cw$
To change from (x, y, z) to
 (ψ, ψ, w) under this change:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 4$$

is transformed (mapped) into:

$$a^2\psi^2 + b^2\psi^2 + c^2w^2 = 4$$

(sphere in (ψ, ψ, w) space)

The plane $z=c$ is mapped into
 $cw=c \Rightarrow w=1$

z plane \leftrightarrow w plane

so the region R in (x, y, z) plane
 is mapped on R' in (u, v, w) space

$$\int \int \int_R dV = \int \int \int_{R'} dz dy dz$$

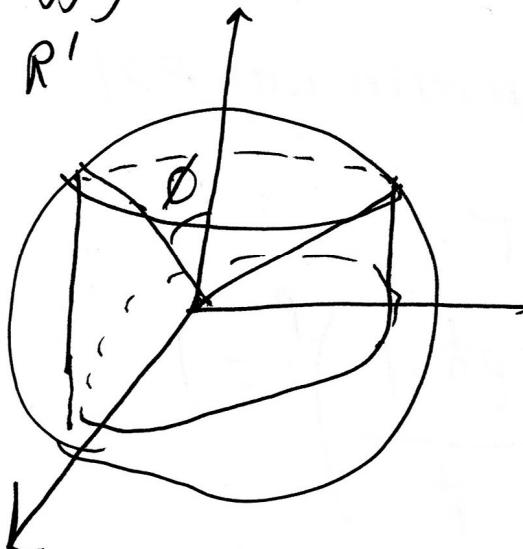
$$\Rightarrow \int \int \int_{R'} |J(u, v, w)| du dv dw$$

Calculate Jacobian:

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\therefore \int \int \int_{R'} abc du dv dw$$



$$① u^2 + v^2 + w^2 = 4$$

in spherical: $\rho = 2$

$$② \text{plane } w = 1 \text{ (w here is } z) \text{ in spherical: } \rho \cos \phi = 1$$

$$③ \text{limits on } \rho \quad \rho = \sec \phi$$

$$④ \sec \phi \leq \rho \leq 2, (0 \leq \theta \leq 2\pi)$$

$$1 \int_0^2 \int_0^{\phi} \dots \therefore \phi = \frac{\pi}{3}$$

$$\therefore 0 \leq \phi \leq \frac{\pi}{3}$$

enter: $\rho = \sec \phi, 0 \leq \phi \leq \frac{\pi}{3}, 0 \leq \theta \leq 2\pi$
 exit: $\rho = 2, \dots, \dots, \dots$

$$\Rightarrow V = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^2 abc \rho^2 \sin \phi d\rho d\phi d\theta$$

$$= abc \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \left(\int_0^2 \left(\frac{\rho^3}{3} \sin \phi \right)^2 \right) d\rho d\phi d\theta$$

$$= abc \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \left(\frac{8 \sin \phi}{3} - \frac{\sin \phi}{3 \cos^3 \phi} \right) d\phi d\theta$$

$$= \frac{abc}{3} \int_0^{2\pi} \left[-8 \cos \phi + 2 \sec^2 \phi \right]_0^{\frac{\pi}{3}} d\theta$$

$$= \frac{11\pi abc}{3}$$

Finals Review Session

$$\ell \left(\frac{n-1}{n+1} \right)^n = \ell \left(\frac{1 - \frac{1}{n}}{1 + \frac{1}{n}} \right)^n = \frac{e^{-1}}{e} = e^{-2}$$

$$\ell \frac{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}}{\ln n^9}$$

(steps are skipped because a similar problem was done before)

$$\Rightarrow \ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} < 1 + \ln n$$

$$\Rightarrow 1 < \ell \frac{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}}{\ln n} < 1 \quad ; \quad \ell \frac{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}}{9 \ln n} = \frac{1}{9}$$

Test for convergence or divergence

$$\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n} \Rightarrow \text{converges by integral test}$$

or
Cauchy condensation test

$$\sum a_n, a_n \downarrow \Rightarrow \text{look at } \sum 2^k a_{2^k} \text{ eg } \sum \frac{1}{n \ln^3 n}$$

$$\Rightarrow \sum 2^k \frac{1}{2^k (\ln 2^k)^3} = \sum \frac{1}{k^3 (\ln 2)^3} \text{ series with } p > 1$$

Find all values of p so that

$$\sum_{n=1}^{\infty} (e^{\frac{1}{n}} - 1)^p \text{ converges}$$

$$\text{solution: } e^{\frac{1}{n}} = 1 + \frac{1}{n} + \frac{1}{2n^2} + \dots$$

$$e^{\frac{1}{n}} - 1 = \frac{1}{n} + \frac{1}{2n^2} + \dots$$

$$\text{Compare with } \sum_{n=1}^{\infty} \left(\frac{1}{n} \right)^p$$

Apply DCT:

$$p = \ell \left(\frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} \right)^p = 1$$

$$p > 1 \quad (\sum \frac{1}{n} \text{ converges})$$

$\therefore \sum (e^{\frac{1}{n}} - 1)^p$ converges for $p > 1$

Written down Taylor series expansion of $f(x) = \frac{x}{2x+3}$ about $x=-1$ and then find the n^{th} derivative of $f(-1)$
 $\Rightarrow f^{(n)}(-1)$ for all n .

$$\begin{aligned}
 & \frac{x}{2x+3} = \frac{x}{2(x+\frac{3}{2})} = \frac{x + \frac{3}{2} - \frac{3}{2}}{2(x+\frac{3}{2})} = \frac{1}{2} - \frac{\frac{3}{2}}{2x+3} \quad z+z^2=4 \\
 & = \frac{1}{2} - \frac{3}{4(x+1)} = \frac{1}{2} - \frac{3}{4(x+1)+6} \quad \text{Method 1:} \\
 & = \frac{1}{2} - \frac{3}{4(x+1)+2} = \frac{1}{2} - \frac{3}{2} \frac{1}{1+2(x+1)} \quad \text{Method 2:} \\
 & = \frac{1}{2} - \frac{3}{2} \sum_{k=0}^{\infty} [-2(x+1)]^k \quad p=2 \quad (1) \\
 & = f(x) = \frac{1}{2} - \frac{3}{2} \sum_{k=0}^{\infty} (-1)^k 2^k (x+1)^k \quad p \cos\theta = p^2 \sin^2\theta \quad (2) \\
 & = \frac{1}{2} + \sum_{k=0}^{\infty} (-1)^{k+1} \cdot 3 \cdot 2^{k-1} (x+1)^k \quad \cos\theta = 2(1-\cos^2\theta) \\
 \text{so } & \frac{f^{(n)}(-1)}{n!} = \text{coefficient of } (x+1)^n \\
 & = (-1)^{n+1} \cdot 3 \cdot 2^{n-1} (x+1)^n
 \end{aligned}$$

$$f(-1) = 0$$

(02 DEC 2016) Lecture 34
 Review #2

Given 2 planes:

$$P_1: 2x+3y-4z=1$$

$P_2: x-y+z=0$
 Find equations of tangent line of intersection.

Find a point (fix $z=0$) [i.e. $2x+3y=1$]

Find gradient of both and

find their cross product.

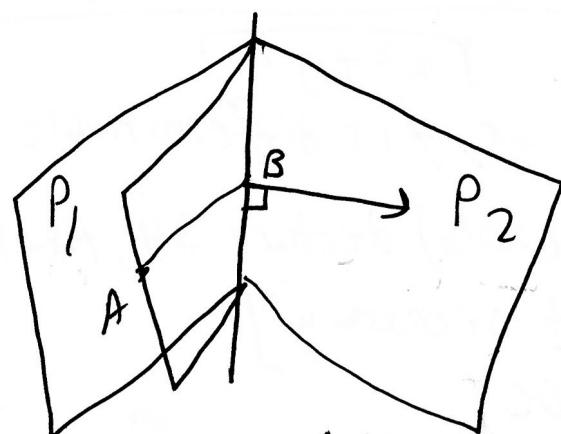
$$x-y=0 \Rightarrow y=x$$

$$2x+3x=1$$

$$5x=1$$

$$x=\frac{1}{5}, y=\frac{1}{5}$$

$$-71- \quad i\left(\frac{1}{5}, \frac{1}{5}, 0\right)$$



Find equation of plane through A & containing the common line of intersection.

Find vector $L (\nabla P_1 \times \nabla P_2)$

then $\overrightarrow{AB} \times L$ is equation of the plane.

2nd method:

P_1 & P_2 are planes

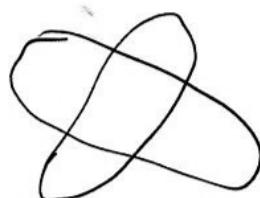
then:

$$P_1 + \lambda P_2 = 0$$

is also a plane containing the common line of intersection

$$i.e. (2x+3y-4z=1) + \lambda(x-y+z)=0$$

Determine λ by $A(1, 1, 2)$
from with ellipses.



$$E_1 + \lambda E_2 = 0$$

Given $f(x, y)$ and a point $(0, 0)$

How do we show f is not continuous?

Either limit doesn't exist or $f(x, y) \neq f(0, 0)$

How do we know f is not differentiable at $(0, 0)$

Having found f is continuous at $(0, 0)$

Find $f_x(0, 0)$ and $f_y(0, 0)$

We look at:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y) - f(0,0) - \Delta x f_x - \Delta y f_y}{\sqrt{x^2+y^2}}$$

If $\ell = 0$, f is differentiable.

(05 DEC 2016) lecture 34, final lecture

[Not necessary]

Prove DCT

If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges
then $\sum_{n=1}^{\infty} a_n$ converges.

$$\textcircled{1} \text{ Let } A_n = \sum_{k=1}^n a_k, B_n = \sum_{k=1}^n b_k$$

Since a_k, b_k are non-negative then

A_n, B_n are increasing

\textcircled{2} Since $a_k \leq b_k$ then $A_n \leq B_n$

\textcircled{3} We are given $\sum_{n=1}^{\infty} b_n$ converges

$\therefore B_n$ converges

$\therefore B_n$ is bounded.

Say $B_n \leq M$ for all $n > N$

$\therefore A_n \leq M$

$\therefore A_n$ is bounded

So A_n being increasing and bounded is convergent

Hence $\sum_{n=1}^{\infty} a_n$ is convergent.

prove DCT

If a_n, b_n are

positive and

$\ell \frac{a_n}{b_n} = \alpha$ where $0 < \alpha < \infty$

and $\sum b_n$ converges, then

$\sum a_n$ converges and vice-versa.

• By existence of $\ell \frac{a_n}{b_n} = L$,

there's an $N > 0$ such that

$\frac{a_n}{b_n} < 2L$ for all $n > N$

Then $a_n < 2Lb_n$, $n \geq N$

This means that

$A_n \leq \text{constant} + 2L B_n$ for all $n \geq N$

e.t.c

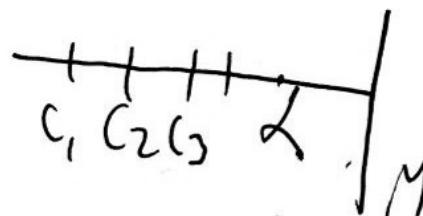
$\Rightarrow B_n \leq M$, $A_n \leq \text{constant} + 2LM$

so A_n is bounded + being increasing \Rightarrow convergent.

Every monotone increasing sequence c_n which is bounded above is convergent.

PROOF:

Say $c_n \leq M$, for all n , by boundedness

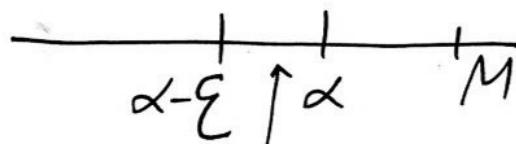


Let α be the least upper bound.

Take $\varepsilon > 0$ and look at $\alpha - \varepsilon$.

$$\int_0^4 \int_0^{\sqrt{4-z}} \frac{z \sin 2z}{4-z} dz dx$$

etc.



There is an element say a_n such that $\alpha - \varepsilon < a_n \leq \alpha$

$$\alpha - \varepsilon < a_n \leq a_n \leq \alpha \text{ for all } n \geq N$$

The axiom of real number system

$$\therefore \alpha - \varepsilon < a_n \leq \alpha \text{ for all } n \geq N$$

$$\Rightarrow |a_n - \alpha| < \varepsilon \text{ for all } n \geq N$$

$$\therefore \ell a_n = \alpha$$

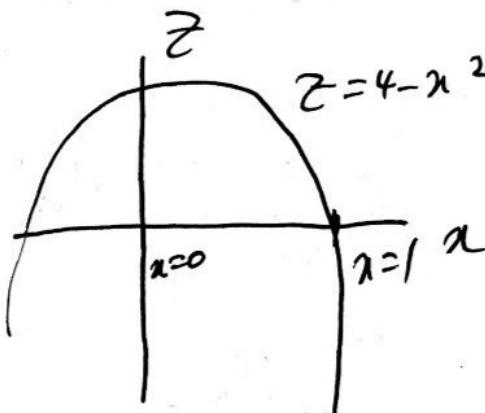
Evaluate:

$$\iiint_D \left(\frac{z \sin 2z}{4-z} \right) dy dz dx$$

0 0 0

$$= \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^z \frac{z \sin 2z}{4-z} dz dx$$

sketch D.



Interchange:

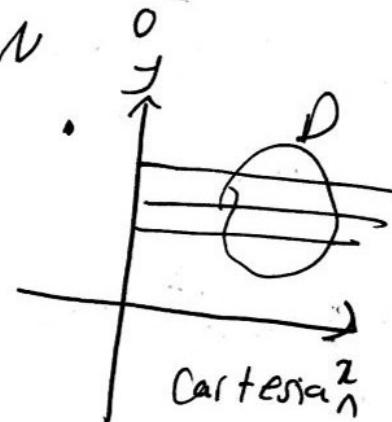
$$\text{enter: } z = 0 \\ \text{exit: } x = \sqrt{4-z} \quad \{ 0 \leq z \leq 4 \}$$

Extra:

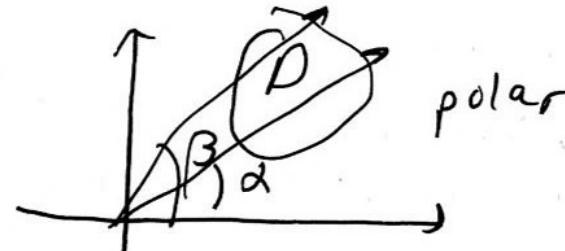
$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$$

Fresnel integrals:

$$\int_0^\infty \sin(x^2) dx$$



Cartesian



polar

Cylindrical:

