

Example:  $f(x, y) = e^{2x} \sin 3y$ , compute  $f_x, f_y, f_{xx}, f_{yy}, \Delta f$

$$f_x = 2e^{2x} \sin 3y \Rightarrow f_{xx} = 4e^{2x} \sin 3y$$

$$f_y = 3e^{2x} \cos 3y \Rightarrow f_{yy} = -9e^{2x} \sin(3y)$$

Also observe  $f_{xy}, f_{yx}$

$$\left. \begin{aligned} f_{xy} &= 6e^{2x} \cos 3y \\ f_{yx} &= 6e^{2x} \cos 3y \end{aligned} \right\} \text{same}$$

$$\Delta f = \text{Laplacian of } f = f_{xx} + f_{yy}$$

$$\therefore \Delta f = 4e^{2x} \sin 3y - 9e^{2x} \sin 3y = -5e^{2x} \sin 3y$$

Challenge Problem:

Let  $f(x, y) = e^{\alpha x} \sin(\beta y)$ . Find  $\alpha$  and  $\beta$  so that  $\Delta f = 0$

Note: when  $\Delta f$  computes to zero, then  $f$  is called a harmonic function. Examples of such functions are:  $e^x \sin y, x, x^2 - y^2, \dots$

(17 OCT 2016) Lecture 17 [05 NOV 2016]

Note on notation:

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} \quad \text{Example: } f(x, y) = e^{\frac{1}{x^2+y^2}} \sin(\pi y)$$

$$\frac{\partial f}{\partial x} = e^{\frac{1}{x^2+y^2}} \cdot \frac{-1}{(x^2+y^2)^2} (2x - y \cos 2y) \text{ etc. ...}$$

• Differentiability of functions of 2 variables:

-  $f$  is differentiable at  $P(x_0, y_0)$  if:

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where  $\epsilon_1, \epsilon_2 \rightarrow 0$  as  $\Delta x \rightarrow 0, \Delta y \rightarrow 0$  and  $\Delta x = x - x_0$   
 $\Delta y = y - y_0$

- To test if  $f$  is differentiable at  $(x_0, y_0)$ :

1<sup>st</sup>: Check continuity at  $(x_0, y_0)$

i.e. if limit doesn't exist or limit exists but  $f(x, y) \neq f(x_0, y_0)$  then  $f$  is NOT continuous at  $(x_0, y_0)$

However if  $f$  is continuous:

2<sup>nd</sup>: compute  $f_x(x_0, y_0)$  &  $f_y(x_0, y_0)$ . If one or both partial's limits don't exist  $\Rightarrow$  not differentiable.

If both exist:

3<sup>rd</sup>: compute the difference:  $f(x, y) - (f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y)$

Then divide the difference by  $\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x-x_0)^2 + (y-y_0)^2}$

Then take limits as  $(x,y) \rightarrow (x_0, y_0)$ . If limit is zero the  $f$  is differentiable at  $(x_0, y_0)$   $\therefore$  the vector  $(f_x, f_y)$  is the "derivative" of  $f$ .

Example:  $f(x,y) = \begin{cases} \frac{x^2 y^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

(i) Is  $f$  continuous?

check  $\lim_{(x,y) \rightarrow (0,0)} f(x,y) \Rightarrow \left| \frac{x^2 y^2}{x^2 + y^2} - 0 \right| \leq \left| \frac{(x^2 + y^2)(x^2 + y^2)}{x^2 + y^2} \right| \leq x^2 + y^2$   
 $\lim_{(x,y) \rightarrow (0,0)} x^2 + y^2 = 0$   
 $\therefore f$  is continuous

(ii) Find  $f_x(0,0)$  and  $f_y(0,0)$  [use the definition]

$\frac{\partial f}{\partial x} \Big|_{(0,0)} = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = \lim_{h \rightarrow 0} 0 = 0$

Similarly  $\frac{\partial f}{\partial y} \Big|_{(0,0)} = 0$

(iii) Is  $f$  differentiable at  $(0,0)$ ?

compute:  $f(x,y) - f(x_0, y_0) - f_x(x_0, y_0) \Delta x - f_y(x_0, y_0) \Delta y$   
 which is  $\frac{x^2 y^2}{x^2 + y^2} - (0)(x) - (0)(y) = \frac{x^2 y^2}{x^2 + y^2}$  ( $f(x_0, y_0) = 0$ , given)

Now divide by  $\sqrt{x^2 + y^2}$

$\Rightarrow \frac{x^2 y^2}{(x^2 + y^2) \sqrt{x^2 + y^2}}$  and calculate its limit.

$\therefore \left| \frac{x^2 y^2}{(x^2 + y^2) \sqrt{x^2 + y^2}} - 0 \right| \leq \left| \frac{(x^2 + y^2)(x^2 + y^2)}{(x^2 + y^2) \sqrt{x^2 + y^2}} \right| \leq \sqrt{x^2 + y^2}$   
 $\lim_{(x,y) \rightarrow (0,0)} \sqrt{x^2 + y^2} = 0$   
 $\therefore f(x,y)$  is differentiable

Note:  $\vec{\nabla} f(0,0) = [f_x(0,0), f_y(0,0)] = (0,0)$

Application: If  $\Delta x, \Delta y$  are small and  $f$  is differentiable at  $(x_0, y_0)$  then we may use the approximation  $f(x,y) \approx f(x_0, y_0) + f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y$ . This is called the linear approximation of  $f$ .

(19 OCT 2016) Lecture 18

Tips: 16.1

$$(x-2)^2 + (y+1)^2 = 4 \Rightarrow \underbrace{x-2 = 2\cos t, y+1 = 2\sin t}_{\text{parametrisation}}$$

So  $\begin{cases} x = 2 + 2\cos t \\ y = 2\sin t - 1 \end{cases}$  and  $t$  depends on the boundary i.e.  $0 \leq t \leq 2\pi$  if full circle.

$$\# \frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow \frac{x}{2} = \cos t, \frac{y}{3} = \sin t$$

$$\therefore \begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases}$$

Find parametrisation of arc of  $y = x^2$  from  $(0,0)$  to  $(2,4)$

Solution:  $\begin{cases} x = t \\ y = t^2 \end{cases} \quad 0 \leq t \leq 2$

Miscellaneous: "gradient"

If  $f(x,y)$  is differentiable at  $(x_0, y_0)$  then  $f$  is continuous at  $(x_0, y_0)$  and has  $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$  as finite numbers.

- To define the derivative:

grad  $f = \vec{\nabla} f = (f_x, f_y)$ . This vector is the derivative

#### 14.4 The chain rule

For 1 variable:

If  $f = f(u)$

$u = u(x)$

$f(u) = f(u(x))$

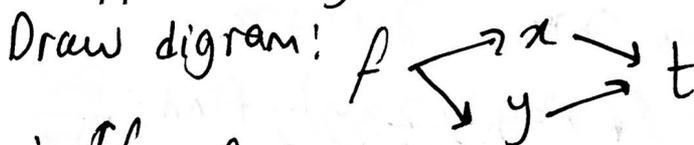
$$\therefore \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

e.g.  $f(x) = \ln(x + e^{x^2})$

$$f'(x) = \frac{1}{x + e^{x^2}} (1 + 2xe^{x^2})$$

For 2 variables:

Suppose  $f(x,y)$  and  $x = x(t), y = y(t)$



$$\therefore \frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

Proof:

$$f(x,y) - f(x_0, y_0) = f_x \Delta x + f_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

Replace  $x = x(t), y = y(t)$

$$\text{Then: } f(x(t), y(t)) - f(x_0(t), y_0(t)) = f_x \Delta x + f_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

Divide both sides by  $\Delta t$ :

$$\therefore \frac{f(x(t), y(t)) - f(x_0(t), y_0(t))}{\Delta t} = f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}$$

Now let  $\Delta t \rightarrow 0$ , then:  $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + (0) \frac{dx}{dt} + (0) \frac{dy}{dt}$

Example:  $f(x, y) = x^2 + y^2$ ,  $x = 3 \cos t$  and  $y = 3 \sin t$

Use  $w$  for  $f(x(t), y(t))$ , then find  $\frac{dw}{dt}$ .

Answer:  $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$  (Chain Rule)

$$\begin{aligned} \frac{dw}{dt} &= 2x(-3 \sin t) + 2y(3 \cos t) \\ &= 2(3 \cos t)(-3 \sin t) + 2(3 \sin t)(3 \cos t) = 0 \end{aligned}$$

Alternatively:

$$w = (3 \cos t)^2 + (3 \sin t)^2 = 9$$

$$\therefore \frac{dw}{dt} = 0$$

\*  $f(x, y)$ ;  $x = x(u, v)$ ,  $y = y(u, v)$

Let  $w = f(x(u, v), y(u, v))$

$\therefore \frac{\partial w}{\partial u}, \frac{\partial w}{\partial v} \Rightarrow$  diagram:  $f \begin{cases} \nearrow x \begin{cases} \nearrow u \\ \searrow v \end{cases} \\ \searrow y \begin{cases} \nearrow u \\ \searrow v \end{cases} \end{cases}$  2 paths

$$\begin{cases} \frac{\partial w}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \\ \frac{\partial w}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} \end{cases}$$

Example 1:  $w = f(x^2 - y^2, 2xy)$ , find  $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$

- Introduce  $u = x^2 - y^2$ ,  $v = 2xy$

Then  $w = f(u, v) \therefore w = f(u, v) \begin{cases} \nearrow u \begin{cases} \nearrow x \\ \searrow y \end{cases} \\ \searrow v \begin{cases} \nearrow x \\ \searrow y \end{cases} \end{cases}$

$$\therefore \frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} (2x) + \frac{\partial f}{\partial v} (2y)$$

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u} (-2y) + \frac{\partial f}{\partial v} (2x)$$

Example 2: Given  $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$ ,  $w = f(x, y)$

(i) Find  $\frac{\partial w}{\partial r}$ ,  $\frac{\partial w}{\partial \theta}$  (ii) Find  $\frac{\partial^2 w}{\partial r^2}$ ,  $\frac{\partial^2 w}{\partial \theta^2}$

Answer:  $\frac{\partial w}{\partial r} = f_x \cos \theta + f_y \sin \theta$

$$\frac{\partial w}{\partial \theta} = f_x (-r \sin \theta) + f_y (r \cos \theta)$$

$$\frac{\partial^2 w}{\partial r^2} = (\cos \theta) \frac{\partial}{\partial r} (f_x) + \sin \theta \frac{\partial}{\partial r} (f_y)$$

etc. ---

Chain rule

(21 OCT 2016) Lecture 19 [Quiz II Review]

1. Let  $f(x) = \int_0^x (\tan^{-1} t) dt$

(i) Find Taylor series generated by  $f$  about  $a=0$

(ii) If  $f(0.1)$  is approximated by  $P_3(0.1)$ , what is the error?

Answers:

(i) First,  $\frac{d}{dt} (\tan^{-1} t) = \frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$

$$\Rightarrow \int_0^x \frac{1}{1+t^2} dt = \sum_{n=0}^{\infty} \int_0^x (-1)^n t^{2n} dt = \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{2n+1} \Big|_0^x$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \text{ so } f(x) = \int_0^x (\tan^{-1} t) dt = \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n+1}}{2n+1} dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)} = \frac{x^2}{1 \cdot 2} - \frac{x^4}{3 \cdot 4} + \frac{x^6}{5 \cdot 6} - \dots$$

(ii)  $P_3(x) = \frac{x^2}{1 \cdot 2}$  (Note: the index of  $P(x)$ , so in this case  $x^3$  doesn't exist, thus  $P_2(x) = P_3(x) = \frac{x^2}{1 \cdot 2}$ )

When we approximate  $f(0.1)$  by  $P_3(0.1)$ , then  $f(0.1) \approx \frac{(0.1)^2}{1 \cdot 2}$

Since the series is strictly alternating,  $|\text{error}| < \text{last unused term}$

$$\therefore |\text{error}| < \frac{(0.1)^4}{3 \cdot 4}$$

2 Let  $f(x) = \sqrt{x}$

- (i) Use Taylor's theorem to find the series of  $f$  about  $a=4$
- (ii) Estimate error when approximating  $f(x)$  by  $P_3(x)$  for  $3 \leq x \leq 5$

Answers:

<u>STEP 1</u>	<u>STEP 2</u>	<u>STEP 3</u>
$f(x) = x^{\frac{1}{2}}$	$f'(4) = 2$	write Taylor series expansion
$f'(x) = (\frac{1}{2})x^{-\frac{1}{2}}$	$f''(4) = (\frac{1}{2})(-\frac{1}{2})$	
$f''(x) = (\frac{1}{2})(-\frac{1}{2})x^{-\frac{3}{2}}$	$f'''(4) = (\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})$	
$f^{(4)}(x) = (\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})x^{-\frac{5}{2}}$	$f^{(4)}(4) = (\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})$	
$f^{(5)}(x) = (\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})x^{-\frac{7}{2}}$	$f^{(5)}(4) = (\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})$	
$f^{(6)}(x) = \dots$	$f^{(6)}(4) = \dots$	

$f(x) = 2 + \frac{(\frac{1}{2})(\frac{1}{2})(x-4)}{1!} + \frac{(\frac{1}{2})(-\frac{1}{2})(x-4)^2}{8 \cdot 2!} + \dots$

(ii) If we approximate  $f(x)$  by  $P_3(x)$ , then error is:

$\frac{f^{(4)}(c)(x-4)^4}{4!}$ ; To estimate error for  $3 \leq x \leq 5$ , bound  $f^{(4)}(c)$

Remember:  $x < c < 4$ ,  $3 \leq x \leq 5$  is  $\frac{3}{x} \frac{4}{c} \frac{5}{x}$   $\therefore 3 < c < 5$

$\Rightarrow f^{(4)}(c) = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{c^{\frac{7}{2}}}$ ; take absolute value,  $|f^{(4)}(c)| = \frac{(\frac{1}{2})(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})}{|c|^{\frac{7}{2}}}$

$\therefore f^{(4)}(c) \leq \frac{(\frac{1}{2})(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})}{3^{\frac{7}{2}}}$  so error  $\leq \frac{(\frac{1}{2})(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})}{3^{\frac{7}{2}} |x-4|^4$

$\leq \frac{(\frac{1}{2})(\frac{1}{2})(\frac{3}{2})(\frac{5}{2}) 1^4}{3^{\frac{7}{2}} 4!}$  3. Expand  $f(x) = \sqrt{x}$  about  $a=4$ , using binomial expansion.

Answer:  $f(x) = \sqrt{x} = \sqrt{(x-4)+4} = \sqrt{4(1 + \frac{x-4}{4})} = 2(1 + \frac{x-4}{4})^{\frac{1}{2}}$   
 $= 2(1 + \frac{(x-4)}{4}(\frac{1}{2}) + \frac{(\frac{1}{2})(-\frac{1}{2})(\frac{x-4}{4})^2}{2!} + \dots)$

4.  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 y^7}{x+y^2}$

Let  $x = -y^2 + y^{15} \therefore \lim_{y \rightarrow 0} \frac{(-y^2 + y^{15})^4 y^7}{y^{15}} = \lim_{y \rightarrow 0} \frac{[y^2(-1+y^{13})]^4 y^7}{y^{15}}$

$= \lim_{y \rightarrow 0} \frac{(-1+y^{13})^4 y^{15}}{y^{15}} = (-1)^4 = 1$

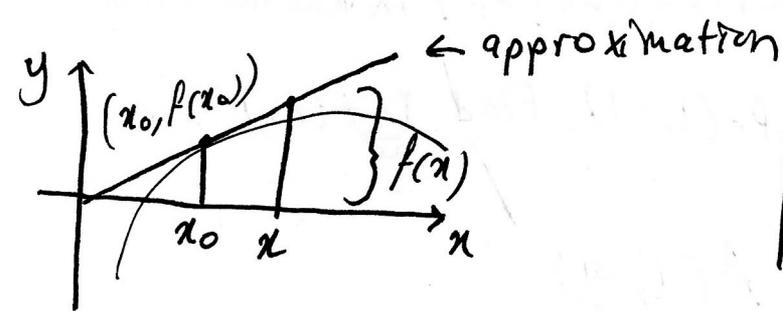
But along  $x=0 \Rightarrow \lim_{y \rightarrow 0} \frac{0}{0+y^2} = 0$  so  $1 \neq 0 \therefore$  limit doesn't exist.

(24 OCT 2016) lecture 20

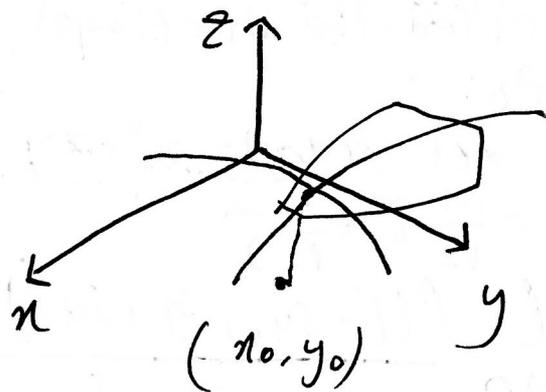
Approximations:

Don't forget:  $\vec{\nabla} f = f_x \vec{i} + f_y \vec{j}$   
= vector

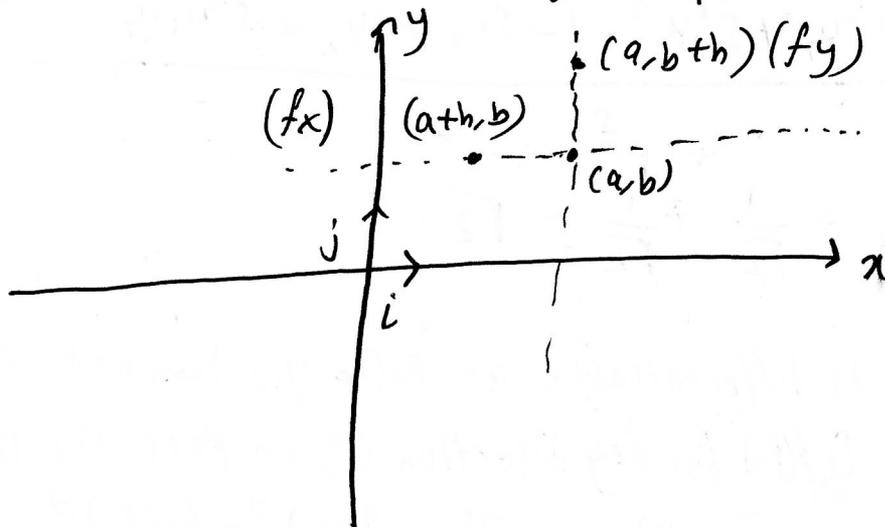
For 1 variable:



For 2 variables:



$f_x(a,b) = \lim_{h \rightarrow 0} \frac{f(a+h,b) - f(a,b)}{h}$ ,  $f_y(a,b) = \lim_{h \rightarrow 0} \frac{f(a,b+h) - f(a,b)}{h}$



- Directional derivatives:

Let  $u = u_1 \vec{i} + u_2 \vec{j}$  be a unit vector, meaning:  $|\vec{u}| = \sqrt{u_1^2 + u_2^2} = 1$

e.g.  $\vec{u} = (\cos \theta) \vec{i} + (\sin \theta) \vec{j} \quad \therefore |\vec{u}| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$

e.g.  $\vec{v} = a \vec{i} + b \vec{j}$ ,  $|\vec{v}| = \sqrt{a^2 + b^2}$  then unit vector =  $\frac{\vec{v}}{|\vec{v}|}$

which is  $\frac{a}{\sqrt{a^2 + b^2}} \vec{i} + \frac{b}{\sqrt{a^2 + b^2}} \vec{j}$

Example: Find a unit vector in the direction from the point  $A(1, 3)$  to  $B(-1, 0)$ .

-  $\vec{AB} = -2\vec{i} - 3\vec{j}$ ,  $|\vec{AB}| = \sqrt{13} \quad \therefore \vec{u} = \frac{-2}{\sqrt{13}} \vec{i} + \frac{-3}{\sqrt{13}} \vec{j}$

Given a function of a point  $P_0(x_0, y_0)$  on a unit vector

$\vec{u} = u_1 \vec{i} + u_2 \vec{j}$ , then  $\lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} = D_{\vec{u}} f(P_0)$

which is called the directional derivative of  $f$  in the direction  $\vec{u}$  at point  $P_0$ .

Example: Let  $f(x, y) = x^2 + xy$ ,  $P_0(1, -1)$ . Find  $D_{\vec{u}} f(P_0)$

where  $\vec{u} = \frac{1}{\sqrt{2}} \vec{i} + \frac{1}{\sqrt{2}} \vec{j}$

Answer:  $\lim_{s \rightarrow 0} \frac{f(1 + su_1, -1 + su_2) - f(1, -1)}{s}$

$$= \lim_{s \rightarrow 0} \frac{(1 + su_1)^2 + (1 + su_1)(-1 + su_2) - (1 - 1)}{s}$$

$$= \lim_{s \rightarrow 0} \frac{1 + 2su_1 + s^2 u_1^2 - 1 - su_1 + su_2 + s^2 u_1 u_2}{s}$$

$$= \lim_{s \rightarrow 0} u_1 + u_2 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

\* Theorem: If  $f$  is differentiable at  $P_0(x_0, y_0)$  then not only does  $f_x$  &  $f_y$  exist but  $D_{\vec{u}} f(P_0)$  for any direction  $\vec{u}$ , in fact the  $D_{\vec{u}} f(P_0) = f_x(P_0) \cdot u_1 + f_y(P_0) \cdot u_2 = \vec{\nabla} f \cdot \vec{u}$

$$\vec{\nabla} f = f_x(P_0) \vec{i} + f_y(P_0) \vec{j}$$

$$\Rightarrow \vec{\nabla} f(P_0) \cdot \vec{u} = f_x(P_0) u_1 + f_y(P_0) u_2$$

Example:  $f(x, y, z) = xy + z \sin x$

Find  $D_{\vec{u}} f(x, y, z)$  at  $(\pi, 1, 0)$ , where  $\vec{u} = \frac{1}{\sqrt{3}} \vec{i} + \frac{1}{\sqrt{3}} \vec{j} + \frac{1}{\sqrt{3}} \vec{k}$

Answer:

$$\left. \begin{aligned} f_x &= y + z \cos x \\ f_y &= x \\ f_z &= \sin x \end{aligned} \right\} \begin{aligned} f_x(P_0) &= 1 + 0 \cos \pi = 1 \\ f_y(P_0) &= \pi \\ f_z(P_0) &= 0 \end{aligned} \quad \therefore D_{\vec{u}} f(P_0) = 1 \cdot \frac{1}{\sqrt{3}} + \pi \left( \frac{1}{\sqrt{3}} \right) + 0 \left( \frac{1}{\sqrt{3}} \right) \\ &= \frac{1 + \pi}{\sqrt{3}}$$

Questions regarding  $\vec{\nabla} f$ ,  $D_{\vec{u}} f$

1. Do these behave like ordinary derivatives with respect to addition, subtraction, multiplication and division?

$$\vec{\nabla}(f+g) \stackrel{?}{=} \vec{\nabla}f + \vec{\nabla}g$$

$$\begin{aligned} & (f+g)_x \vec{i} + (f+g)_y \vec{j} \\ &= (f_x + g_x) \vec{i} + (f_y + g_y) \vec{j} \\ &= \underbrace{f_x \vec{i} + f_y \vec{j}}_{\vec{\nabla}f} + \underbrace{g_x \vec{i} + g_y \vec{j}}_{\vec{\nabla}g} \end{aligned}$$

$$\vec{\nabla}(fg) = ?$$

$$\begin{aligned} \vec{\nabla}(fg) &= \frac{\partial}{\partial x}(fg) \vec{i} + \frac{\partial}{\partial y}(fg) \vec{j} \\ &= \left( f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x} \right) \vec{i} + \left( f \frac{\partial g}{\partial y} + g \frac{\partial f}{\partial y} \right) \vec{j} \\ &= f \left( \frac{\partial g}{\partial x} \vec{i} + \frac{\partial g}{\partial y} \vec{j} \right) + g \left( \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} \right) \\ &= f \vec{\nabla}g + g \vec{\nabla}f \end{aligned}$$

• Exercise to do by yourself:  $\vec{\nabla} \left( \frac{f}{g} \right)$

- For  $D_{\vec{u}}(f+g)$ :

$$= \vec{\nabla}(f+g) \cdot \vec{u} = (\vec{\nabla}f + \vec{\nabla}g) \cdot \vec{u} = \vec{\nabla}f \cdot \vec{u} + \vec{\nabla}g \cdot \vec{u} = D_{\vec{u}}f + D_{\vec{u}}g$$

2. How large can  $D_{\vec{u}} f(P_0)$  be? and in what direction?

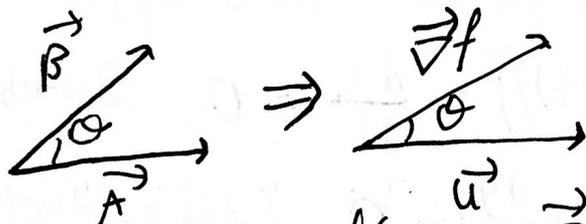
Answer: Recall dot product of 2 vectors:

$$\vec{A} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}; \quad \vec{B} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$\text{then } \vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (\text{analytic})$$

Geometrically:

$$\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cos \theta$$



$$\therefore D_{\vec{u}} f(P_0) = \vec{\nabla}f(P_0) \cdot \vec{u} \\ = |\vec{\nabla}f(P_0)| \cdot |\vec{u}| \cos \theta$$

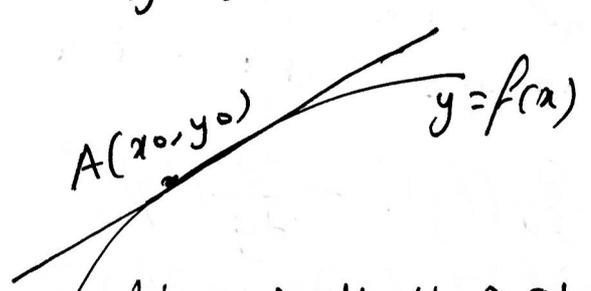
If  $|\vec{u}|=1$  then  $|\vec{\nabla}f(p_0)| \cos \theta \therefore$  largest  $D_{\vec{u}}f(p_0) = |\vec{\nabla}f(p_0)|$   
 that is, when  $\theta=0$ , or when  $\vec{u}$  is in the same direction as the gradient of  $\vec{\nabla}f$ .

Similarly the smallest  $D_{\vec{u}}f(p_0) = -|\vec{\nabla}f(p_0)|$  ( $\theta=\pi$ , or when  $\vec{u}$  is in the opposite direction of  $\vec{\nabla}f$ )

(26 OCT 2016) lecture 21

Let us go back to the Brevet years :)

- Finding equation of tangent to curve at A.

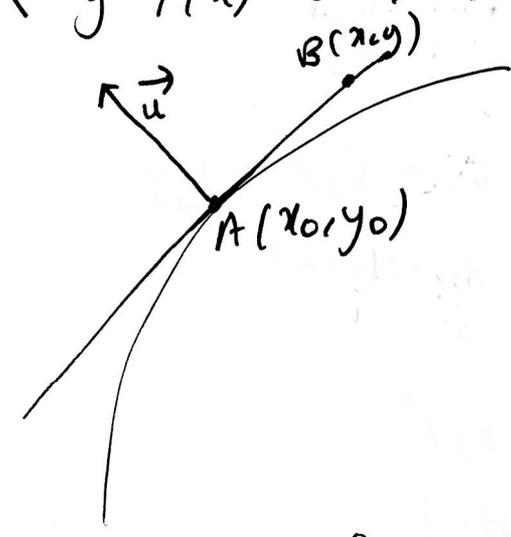


- ① Find  $\frac{dy}{dx}$
- ② Find  $\frac{dy}{dx}$  at  $A(x_0, y_0) \Rightarrow$  gradient
- ③  $y - y_A = m(x - x_A)$  DONE!

Now in Math 201...

- Given a function of 2 variables  $F(x, y)$ , a level curve  $C$  of  $f$  (i.e.  $F(x, y) = c$ ) and a point on the level curve  $(x_0, y_0)$ , how to find equation of the tangent line to  $C$  at  $(x_0, y_0)$

\*  $y - f(x) = 0 \Rightarrow F(x, y) = 0$



If we can find a vector  $\vec{u}$  of A orthogonal to the vector  $\vec{AB}$  then:  $\vec{AB} \cdot \vec{u} = 0$

- Given a level curve  $F(x, y) = c$  &  $A(x_0, y_0)$  on it, find a vector normal to the curve at A. To see what this normal is, let the curve be parametrised by:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad a < t < b$$

$F(x(t), y(t)) = c$  for all  $t$ . Now differentiate

$$\frac{d}{dt}(F(x(t), y(t))) = \frac{dc}{dt} = 0. \text{ But by chain rule:}$$

$$\Rightarrow F_x \frac{dx}{dt} + F_y \frac{dy}{dt} = 0 \quad (\text{dot product})$$

i.e.  $(F_x \vec{i} + F_y \vec{j}) \cdot \left( \frac{dx}{dt} \vec{i} + \frac{dy}{dt} \vec{j} \right) = 0$  \*  $\vec{v}$ : velocity vector

$\therefore \vec{\nabla}f(x_0, y_0)$  is perpendicular to curve.

Problem: Find equation of tangent line to  $\frac{x^2}{4} + y^2 = 2$  at  $(-2, 1)$

Solution: The given curve is a level curve of function of 2 variables

$$F(x, y) = \frac{x^2}{4} + y^2.$$

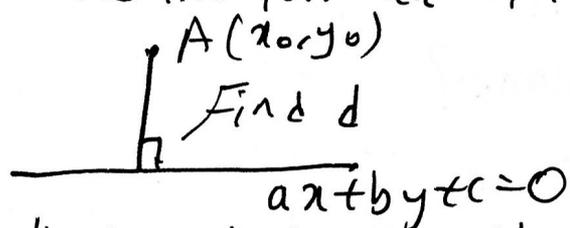
Find gradient:

$$\vec{\nabla}f(x_0, y_0) = \frac{x}{2} \vec{i} + 2y \vec{j}; \text{ at } (-2, 1), \vec{\nabla}f = -\vec{i} + 2\vec{j}$$

$\therefore$  Equation of tangent line is  $(x+2)(-1) + (y-1)(2) + (z-0)(0) = 0$

Challenge Problem:

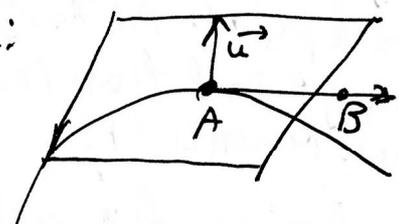
Given a line  $ax + by + c = 0$ , and a point outside the line A, what is the formula of the shortest distance from A to the line?



- Now with 3 variables:

Given a function of 3 variables  $F(x, y, z)$  and a level surface  $S$

$F(x, y, z) = c$  and a point  $A(x_0, y_0, z_0)$  on  $S$ , want to find equation of tangent plane to  $S$  at  $A$ . i.e:



- let  $x = x(t), y = y(t), z = z(t)$  be parametric equations of a curve  $C$

on the surface  $S$ , passing through the point  $A$ . Then

$$F(x(t), y(t), z(t)) = c; \frac{d}{dt}(F(x(t), y(t), z(t))) = \frac{dc}{dt} = 0$$

But, according to chain rule:

$$\frac{d}{dt}(F(x(t), y(t), z(t))) = F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt}$$

$$\Rightarrow (F_x \vec{i} + F_y \vec{j} + F_z \vec{k}) \cdot \left( \frac{dx}{dt} \vec{i} + \frac{dy}{dt} \vec{j} + \frac{dz}{dt} \vec{k} \right) = 0$$

$$\vec{\nabla}f(A) \cdot \vec{v} = 0$$

$\therefore \vec{\nabla}f(x_0, y_0, z_0)$  is a vector orthogonal to  $S$  at  $A$  then the equation of tangent plane to  $S$  at  $A$  is:

$$(x-x_0)f_x + (y-y_0)f_y + (z-z_0)f_z = 0$$

Example: Find equation of tangent plane to the graph of  $z = x^2 + y^2$  at the point  $(-1, 1, 2)$

①  $z - x^2 - y^2 = 0$  ← level surface

$F(x, y, z) = 0$

②  $\vec{\nabla} f(x, y, z) = -2x\vec{i} - 2y\vec{j} + \vec{k}$ ,  $\vec{\nabla} f(-1, 1, 2) = 2\vec{i} - 2\vec{j} + \vec{k}$

③ Equation:  $2(x+1) - 2(y-1) + (z-2) \cdot 1 = 0$

- For the same surface find equation of the normal line along A

$\Rightarrow \begin{cases} x+1 = 2t \\ y-1 = -2t \\ z-2 = 1t \end{cases} \quad -\infty < t < +\infty$

Where does this line intersect the  $xz$  plane?

-  $xz$ -plane  $\Rightarrow y=0 \therefore t = \frac{1}{2}$  etc —

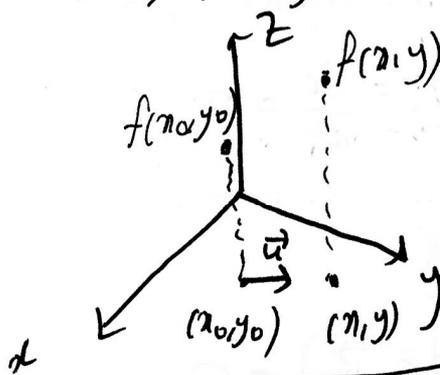
(28 OCT 2016) lecture 22

- Change in a given direction:

Given  $f(x, y)$  and a point  $A(x_0, y_0)$ ,  $\vec{u}$  (unit vector). If we consider the change in  $f$  from  $(x_0, y_0)$  to  $(x, y)$ , how much is it and approximately how much is it.

- Change in  $f \Rightarrow f(x, y) - f(x_0, y_0) \approx D_{\vec{u}} f(x_0, y_0) ds$

Distance between  $(x, y)$  &  $(x_0, y_0)$



So approximation:

$f(x, y) = f(x_0, y_0) + f_x(\Delta x) + f_y(\Delta y)$

- how large can the error be?

[07 NOV 2016]

Answer: If  $f_x, f_y, f_{xy}$  are all continuous in a region containing  $(x_0, y_0)$  & if  $|f_x| \leq M, |f_y| \leq M, |f_{xy}| \leq M$  at all points, then error:  $|\text{error}| \leq \frac{1}{2} M [(x-x_0) + (y-y_0)]^2$

Note: The derivation will be linked to Taylor series.

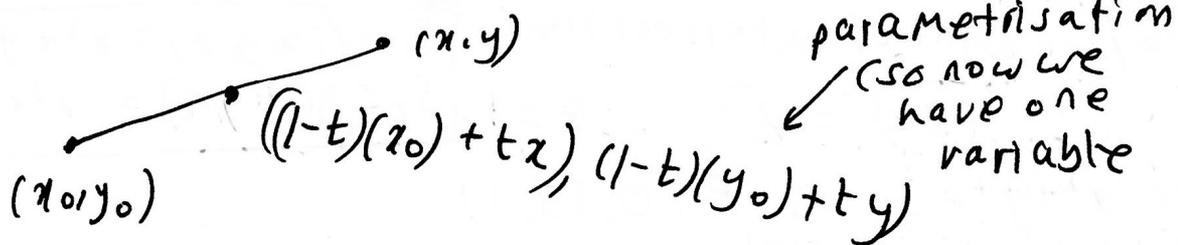
Derivation:

$f(t) = f(t_0) + f'(t_0)(t - t_0) + \frac{f''(c)}{2!} (t - t_0)^2$

If  $f(t)$  is approximated by  $f(t_0) + f'(t_0)(t - t_0)$ , and if  $|f''(c)| \leq M$ , then  $|\text{error}| \leq \frac{1}{2} M (t - t_0)^2$  (for one variable)

So now we need Taylor's theorem for 2 variables.

Problem: Given  $f(x, y)$  and a point  $(x_0, y_0)$ , to find a Taylor series expansion of  $f$  about  $(x_0, y_0)$ , introduce auxiliary of  $f$  of 1 variable.



$$F(t) = f((1-t)x_0 + tx, (1-t)y_0 + ty) \quad 0 \leq t \leq 1$$

$$F(1) = F(0) + (1-0)F'(0) + \frac{(1-0)^2}{2!} F''(c) \quad 0 < c < 1$$

$$\therefore f(x, y) = f(x_0, y_0) + \text{chain rule} \downarrow \left[ F'(t) = f_x(x-x_0) + f_y(y-y_0) \right] + \frac{1}{2} [F''(t)]^*$$

$$* F''(t) = [(x-x_0) f_{xx}(x-x_0) + f_{xy}(x-x_0)(y-y_0) + (y-y_0) f_{yy}(y-y_0) + f_{yx}(y-y_0)(x-x_0)]$$

$$F''(t) = \frac{1}{2} [ (x-x_0)^2 f_{xx}(c_1, c_2) + 2(x-x_0)(y-y_0) f_{xy} + (y-y_0)^2 f_{yy}(c_1, c_2) ]$$

If the partial derivatives are less than  $M$ , then  $M$  is a common factor so  $F''(t) < \frac{1}{2} M [(x-x_0)^2 + (y-y_0)^2]$

Note:  $\frac{d}{dt} \left( \frac{\partial f}{\partial x} \right) = f_{xx} \frac{dx}{dt} + f_{xy} \frac{dy}{dt}$  Factorised  
 $[f_x \langle y \rangle t]$

-compute  $\vec{\nabla} f$  if  $f(x, y) = x^2 e^y + \sin x$

$$\vec{\nabla} f = (2x e^y + \cos x) \vec{i} + (x^2 e^y) \vec{j}$$

-compute  $\vec{\nabla} \cdot \vec{\nabla} f$  ( $\vec{\nabla}(\text{del}) = \frac{\partial}{\partial x} (\ ) \vec{i} + \frac{\partial}{\partial y} (\ ) \vec{j}$ )

$$\begin{aligned} \therefore \vec{\nabla} \cdot \vec{\nabla} f &= \left( \frac{\partial}{\partial x} (\ ) \vec{i} + \frac{\partial}{\partial y} (\ ) \vec{j} \right) \cdot (f_x \vec{i} + f_y \vec{j}) \\ &= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = f_{xx} + f_{yy} = \Delta f \end{aligned}$$

$\Delta$  Laplacian

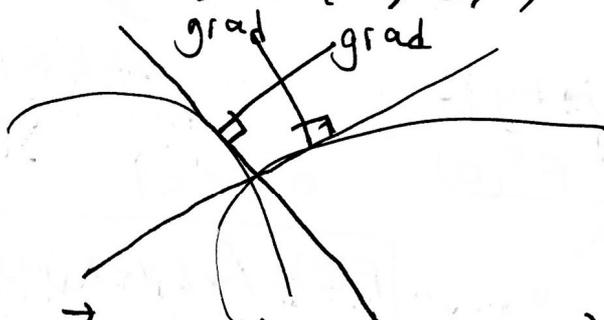
Hence  $\vec{\nabla} \cdot \vec{\nabla} f = 2e^y \sin z + x^2 e^y$

• Let  $S_1$  &  $S_2$  be the surface given by  $x^2 + y^2 + (z-2)^2 = 4$ . Find equations of tangent line to the intersection of  $S_1$  &  $S_2$  at point  $A(\sqrt{3}, 0, 1)$

- To find point of intersection:  $S_1 = f(x, y, z) = x^2 + y^2 + (z-2)^2 = 4$   
 $S_2 = g(x, y, z) = x^2 + y^2 + z^2 = 4$   
 $(z-2)^2 = z^2 \implies z = 1$

So  $x^2 + y^2 = 3 \implies B(0, \sqrt{3}, 1)$

Sketch:



Take cross product of the gradients.

\*  $\vec{\nabla} f = 2x\vec{i} + 2y\vec{j} + 2(z-2)\vec{k}$ ;  $\vec{\nabla} f(A) = 2\sqrt{3}\vec{i} - 2\vec{k}$  (normal to  $S_1$  at  $A$ )  
 \*  $\vec{\nabla} g = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$ ;  $\vec{\nabla} g(A) = 2\sqrt{3}\vec{i} + 2\vec{k}$  (normal to  $S_2$  at  $A$ )

\*  $\vec{\nabla} f(A) \times \vec{\nabla} g(A) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\sqrt{3} & 0 & -2 \\ 2\sqrt{3} & 0 & 2 \end{vmatrix} = -8\sqrt{3}\vec{j}$

$\therefore$  parametric equation of tangent line:

$$\begin{cases} x - \sqrt{3} = 0, t \\ y - 0 = -8\sqrt{3}t \\ z - 1 = 0, t \end{cases} \implies \begin{cases} x = \sqrt{3} \\ y = -8\sqrt{3}t \\ z = 1 \end{cases}$$

(02 NOV 2016) Lecture 23

Extreme Values (Maxima & Minima)

Given  $f(x, y)$  defined on some domain on the  $x$ - $y$  plane:

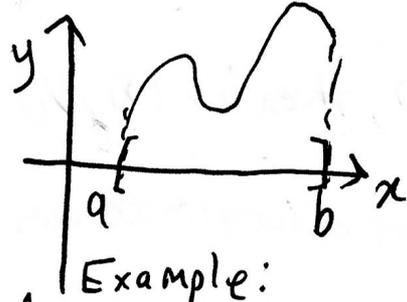
- 1) Find all points that are critical points
- 2) Decide which is a local minimum, a local maximum or a saddle point.
- 3) Find, if possible, the absolute maximum & minimum of  $f$  on  $D$ .

- Domain:



• In one variable:

$f(x)$  is defined on an interval  $a \leq x \leq b$ .



To find extreme values of  $f$ :

1) Compute  $f'(x)$ , set it equal to zero. Then solve for  $x$ , then get critical points,  $x_1, x_2, \dots$

Example:

$f(x) = x^3 - 2x + 1 \quad 0 \leq x \leq 10$

$f'(x) = 3x^2 - 2$ ;  $f'(x) = 0$  so,  $3x^2 - 2 = 0 \therefore x = \pm \sqrt{\frac{2}{3}} \therefore 2$  critical points

Important to check the end points:

Consider end points  $a$  and  $b$

- Compute  $f(a), f(x_1), f(x_2), \dots, f(b)$ . The largest is local maximum and the lowest the local minimum.
- locally compute  $f''(x)$  at critical points and decide for local minima or maxima.

•  $x = \pm \sqrt{\frac{2}{3}}$  •  $f''(x) = 6x$  •  $f(0) = 1$   
 $0, \sqrt{\frac{2}{3}}, 0$  •  $\sqrt{\frac{2}{3}}$  is a local minimum •  $f(\sqrt{\frac{2}{3}}) = \dots$

• In 2 variables:

To find critical points of  $f(x, y)$ :

- ① Compute  $f_x, f_y$  and set it equal to zero. (i.e.  $f_x(x, y) = 0$   
 $f_y(x, y) = 0$ )  
 Solve 2 equations 2 unknowns, then get critical points  $(x_1, y_1), (x_2, y_2), \dots$
- ② Compute  $f_{xx}, f_{yy}, f_{xy}$  and evaluate them at critical points. (But which should we use?)

Intermission:

By Taylor's theorem for  $f(x, y)$

$$f(x, y) = f(x_1, y_1) + (x - x_1)f_x(x_1, y_1) + (y - y_1)f_y(x_1, y_1) + \frac{1}{2!} \left\{ (x - x_1)^2 f_{xx}(c, d) + 2(x - x_1)(y - y_1)f_{xy}(c, d) + (y - y_1)^2 f_{yy}(c, d) \right\}$$

But  $f_x(x_1, y_1)$  &  $f_y(x_1, y_1) = 0 \therefore f(x, y) = f(x_1, y_1) + \boxed{Ax^2 + 2BXY + CY^2}$

where  $A = f_{xx}, B = f_{xy}, C = f_{yy}, X = (x - x_1), Y = (y - y_1)$

\*  $Y^2 \left( \frac{AX^2}{Y^2} + \frac{2BX}{Y} + C \right) = Y^2 [Au^2 + 2Bu + C]$  where  $u = \frac{X}{Y}$

Apply quadratic formula:  $\Delta = (2B)^2 - 4AC = 4(B^2 - AC)$

So if  $\Delta < 0$ , same sign as  $A$

② contd

If  $B^2 - AC < 0$ ,  $(f_{xy})^2 - f_{xx}f_{yy} < 0$  and  $f_{xx} > 0$ , then at  $(x_1, y_1)$  the point is a local minimum

If  $(f_{xy})^2 - f_{xx}f_{yy} < 0$  and  $f_{xx} < 0$ , then  $(x_1, y_1)$  is a point of local maximum

If  $(f_{xy})^2 - f_{xx}f_{yy} > 0$ , then  $(x_1, y_1)$  is a saddle point.

If  $(f_{xy})^2 - f_{xx}f_{yy} = 0$ , cannot decide.

③ Boundary: eg  $\triangle$ ,  $\square$ ,  $\circ$  etc.

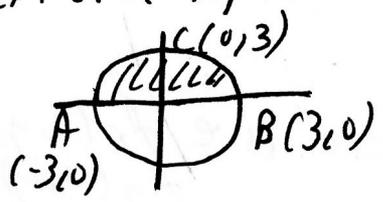
• Parametrise the boundary i.e.  $x = x(t)$ ,  $y = y(t)$ ,  $a \leq t \leq b$

• Then use these in  $f(x, y)$  to get  $o$  or  $y = g(x)$   $c \leq x \leq d$  a function of 1 variable.

• Let  $f(x, y) = x^2 + y^2 - 3x - 2y + 10$  defined on  $D = \{(x, y) \mid x^2 + y^2 \leq 9, y > 0\}$

Find all local extrema of  $f$  and absolute extrema of  $f$  on  $D$ .

- Sketch  $D$ :



- Boundary of  $D$ :

$AB$ :  $y = 0$   $(-3 \leq x \leq 3)$ ;  $BCA$ :  $y = +\sqrt{9 - x^2}$   $(-3 \leq x \leq 3)$

• Compute  $f_x$   $f_y$

$f_x = 2x - 3$ ,  $f_y = 2y - 2$ ;  $f_x = 0 \Rightarrow 2x - 3 = 0$ ,  $x = \frac{3}{2}$  } 1 critical

• Check if this point is in  $D$ :  $f_y = 0 \Rightarrow 2y - 2 = 0$ ,  $y = 1$  } point  $(\frac{3}{2}, 1)$

$(\frac{3}{2})^2 + 1^2 \leq 9$  (verified)

•  $f_{xx} = 2$ ,  $f_{xy} = 0 \Rightarrow (f_{xy})^2 - f_{xx}f_{yy} = 0 - 4 = -4 < 0$

$f_{yy} = 2$  But  $f_{xx} > 0$  so the point is a local maximum.

• Boundary: -  $AB$ :  $y = 0$ ,  $f(x, 0) = g(x)$

-  $BCA$ :  $y = +\sqrt{9 - x^2}$   $= x^2 + 3x + 10$   $(-3 \leq x \leq 3)$

So  $h(x) = f(x, \sqrt{9 - x^2}) = 9 - 3x - 2\sqrt{9 - x^2} + 10 = 19 - 3x - 2\sqrt{9 - x^2}$

To be continued